AN APPLICATION OF GENERALIZED JACOBI TRANSFORM TO SOLVE A PARTIAL DIFFERENTIAL EQUATION GOVERNING HEAT CONDUCTION IN A MOVING ANISOTROPIC RECTANGULAR SLAB

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In the present paper unsteady heat distribution in a moving anisotropic slab has been discussed by using the generalized Jacobi transform recently introduced by Sharma. In the two cases one involving finite and the other infinite one, the slab moves with a constant velocity and contains a continuous source of heat situated inside it. The temperatures of all the four vertical faces are known and initial temperature of the solid is given.

INTRODUCTION

The problems dealing with heat conduction in anisotropic solids are of practical importance. Such problems arise in considering the conduction in crystals, rocks, wood and laminated materials such as transformer cores, etc. Carslaw and Jaeger (1959), Gupta and Saxena (1970, 1971) suggested a few problems of these solids treated under certain conditions.

In this paper an attempt has been made to solve a partial differential equation by an application of generalized Jacobi transform governing unsteady heat conduction in solids of given thickness, having both finite and infinite dimensions and moving with a constant velocity along the direction of one of its horizontal edges. The temperature of all the four vertical faces are supposed to be known and the initial temperature of the solid is given. The slab is supposed to be formed of very thin laminar layers parallel to the horizontal plane, so that the variation of the conductivity occurs only in the direction of the vertical edge.

EQUATION AND BOUNDARY CONDITIONS

The partial differential equation governing heat conduction in a solid moving along the direction of x-axis with a constant velocity $-U$ is

$$\frac{\partial}{\partial x}\left[K_x \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y \frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[K_z \frac{\partial u}{\partial z}\right] - U\frac{\partial u}{\partial x} + Q(x, y, z, t) = c' \frac{\partial u}{\partial t} \qquad \ldots(1)$$

where K_x, K_y and K_z are conductivities along the directions of principal axes and $Q(x, y, z, t)$ is intensity of a continuous source of heat situated at the point (x, y, z).

We consider two slabs $x \geqslant 0, y \geqslant 0, c \leqslant z \leqslant d$ and $0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, c \leqslant z \leqslant d$.

Also we assume $K_x = K_y = \lambda,\ K_z = \lambda(z-c)(d-z)$, ...(2)

where λ is a constant and independent of x, y, z and t.

The mathematical statement of the conditions is

$$u(0, y, z, t) = \theta(y, z, t) \qquad \text{...(3)}$$

$$u(x, 0, z, t) = \phi(x, z, t) \qquad \text{...(4)}$$

$$u(x, y, 0, t) = 0 \qquad \text{...(5)}$$

$$u(x, y, z, 0) = \eta(x, y, z). \qquad \text{...(6)}$$

(i) *Infinite slab*

$$\lim_{x\to\infty} u(x, y, z, t) = \lim_{y\to\infty} u(x, y, z, t) = 0. \qquad \text{...(7)}$$

(ii) *Finite slab*

$$u(a, y, z, t) = 0,\ u(x, b, z, t) = \sigma(x, z, t). \qquad \text{...(8)}$$

There is no radiation from the faces $z = c$ and $z = d$.

Solution of the Equation

We have to solve the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial z}\left[(z-c)(d-z)\frac{\partial u}{\partial z}\right] - \frac{U}{\lambda}\frac{\partial u}{\partial x} + \frac{1}{\lambda}Q(x, y, z, t) = \frac{c'}{\lambda}\frac{\partial u}{\partial t} \qquad \text{...(9)}$$

associated with the above conditions.

First, we shall use generalized Jacobi transform defined by Sharma (1972) as

$$u_n(x, y, t) = \int_c^d u(x, y, z, t)\, P_n^{(\alpha,\beta)}(z)\, dz \qquad \text{...(10)}$$

and having the inversion formula

$$u(x, y, z, t) = \sum_{n=0}^{\infty} \frac{(z-c)^\beta (d-z)^\alpha}{\delta_n}\, u_n(x, y, t)\, P_n^{(\alpha,\beta)}(z) \qquad \text{...(11)}$$

where

$$\delta_n = \frac{(d-c)^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}. \qquad \ldots(12)$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is the solution of the differential equation

$$\frac{d}{dz}\left[(z-c)(d-z)\,\frac{dP_n^{(\alpha,\beta)}(z)}{dz}\right] + \{(d\beta + c\alpha) - (\alpha+\beta)\,z\}\,\frac{dP_n^{(\alpha,\beta)}(z)}{dz} + n(\alpha+\beta+n+1)\,P_n^{(\alpha,\beta)}(z) = 0 \qquad \ldots(13)$$

where

$$P_n^{(\alpha,\beta)}(z) = \frac{(z-c)^{-\beta}\,(z-d)^{-\alpha}}{n!\,(d-c)^n}\,\frac{d^n}{dz^n}\left[(z-c)^{\beta+n}\,(z-d)^{\alpha+n}\right],$$
$$(\alpha, \beta > -1,\ d > c). \qquad \ldots(14)$$

Using (13) and taking generalized Jacobi transform (10) and using the condition (5), eqn. (9) becomes

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} - \frac{U}{\lambda}\,\frac{\partial u_n}{\partial t} - (n+1)\,(n+\alpha+\beta)\,u_n = -\frac{1}{\lambda}\,Q(x, y, t) + \frac{c'}{\lambda}\,\frac{\partial u_n}{\partial t}. \qquad \ldots(15)$$

The other conditions are transformed as

$$u_n(0, y, t) = \theta_n(y, t) \qquad \ldots(16)$$

$$u_n(x, 0, t) = \phi_n(x, t) \qquad \ldots(17)$$

$$u_n(x, y, 0) = \eta_n(x, y) \qquad \ldots(18)$$

$$\lim_{x\to\infty} u_n(x, y, t) = \lim_{y\to\infty} u_n(x, y, t) = 0 \qquad \ldots(19)$$

$$u_n(a, y, t) = 0,\quad u_n(x, b, t) = \sigma_n(x, t). \qquad \ldots(20)$$

Now, we shall use the Fourier sine transforms

$$u_{ns}(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin Ay\; u_n(x, y, t)\, dy \qquad \ldots(21)$$

and

$$u_{ns}(x, t) = \int_0^b \sin \frac{m\pi y}{b}\; u_n(x, y, t)\, dy \qquad \ldots(22)$$

according as the slab is infinite or finite.

The inversion formulae of the above transforms are given as

$$u_n(x, y, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin Ay \, u_{ns}(x, t) \, dA \qquad \ldots(23)$$

and

$$u_n(x, y, t) = \frac{2}{b} \sum_{m=1}^{\infty} \sin \frac{m\pi y}{b} u_{ns}(x, t) \qquad \ldots(24)$$

respectively.

Taking infinite and finite sine transforms of eqn. (15), we get

$$\frac{\partial^2 u_{ns}}{\partial x^2} - \frac{U}{\lambda} \frac{\partial u_{ns}}{\partial x} - [A^2 + (n+1)(n+\alpha+\beta)] \, u_{ns}(x, t)$$
$$= \frac{c'}{\lambda} \frac{\partial u_{ns}}{\partial t} + A \sqrt{\frac{2}{\pi}} \phi_n(x, t) - \frac{1}{\lambda} Q_{ns}(x, t) \qquad \ldots(25)$$

and

$$\frac{\partial^2 u_{ns}}{\partial x^2} - \frac{U}{\lambda} \frac{\partial u_{ns}}{\partial x} - \left[\frac{m^2\pi^2}{b^2} + (n+1)(n+\alpha+\beta)\right] u_{ns}(x, t)$$
$$= \frac{c'}{\lambda} \frac{\partial u_{ns}}{\partial t} - \frac{m\pi}{b} [(-1)^{m+1} \sigma_n(x, t) + \phi_n(x, t)] - \frac{1}{\lambda} Q_{ns}(x, t) \qquad \ldots(26)$$

respectively, where

$$Q_{ns}(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin Ay \, Q_n(x, y, t) \, dy \qquad \ldots(27)$$

$$Q_{ns}(x, t) = \int_0^b \sin \frac{m\pi y}{b} Q_n(x, y, t) \, dy. \qquad \ldots(28)$$

Now, we shall use the well-known Laplace transform defined as

$$L\{h(t)\} = \int_0^\infty e^{-pt} h(t) \, dt \qquad \ldots(29)$$

and its inversion formula gives

$$h(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} L\{h(t)\} \, dp. \qquad \ldots(30)$$

Also, we have

$$L\left\{\frac{\partial^n h}{\partial t^n}\right\} = p^n L\{h(t)\} - p^{n-1} h(0) - p^{n-2} h'(0) \ldots \qquad \ldots(31)$$

Taking Laplace transforms of eqns. (25) and (26), we obtain

$$\frac{d^2 v}{dx^2} - \frac{U}{\lambda}\frac{dv}{dx} - \left[A^2 + (n+1)(n+\alpha+\beta) + \frac{c'p}{\lambda}\right] v(x, p) = t(x) \quad \ldots(32)$$

and

$$\frac{d^2 v'}{dx^2} - \frac{U}{\lambda}\frac{dv'}{dx} - \left[\frac{m^2\pi^2}{b^2} + (n+1)(n+\alpha+\beta) + \frac{c'p}{\lambda}\right] v'(x, p) = t'(x) \qquad \ldots(33)$$

where

$$v(x, p) = L\{u_{ns}(x, t)\},\ v'(x, p) = L\{u_{ns}(x, t)\} \qquad \ldots(34)$$

$$t(x) = A\sqrt{\frac{2}{\pi}}\, L\{\phi(x, t)\} - \frac{1}{\lambda}\, L\{Q_{ns}(x, t)\} - \frac{c'}{\lambda}\eta_{ns}(x) \qquad \ldots(35)$$

and

$$t'(x) = \frac{m\pi}{b}(-1)^m L\{\sigma_n(x, t)\} - \frac{m\pi}{b} L\{\phi_n(x, t)\} - \frac{1}{\lambda} L\{Q_{ns}(x, t)\} - \frac{c'}{\lambda}\eta_{ns}(x). \qquad \ldots(36)$$

The set of conditions associated with the differential eqns. (32) and (33) are

$$\text{(i)}\ \ v(0, p) = \zeta(p); \quad \text{(ii)}\ \ v'(0, p) = \zeta'(p),\ v'(a, p) = 0 \qquad \ldots(37)$$

where

$$\zeta(p) = L\left\{\int_0^\infty \sin Ay \int_c^d P_n^{(\alpha,\beta)}(z)\, \theta(y, z, t)\, dz\, dy\right\} \qquad \ldots(38)$$

$$\zeta'(p) = L\left\{\int_0^b \sin\frac{m\pi y}{b} \int_c^d P_n^{(\alpha,\beta)}(z)\, \theta(y, z, t)\, dz\, dy\right\}, \qquad \ldots(39)$$

and

$$\frac{m^2\pi^2}{b^2} + (n+1)(n+\alpha+\beta) = B. \qquad \ldots(40)$$

Now solving the ordinary differential equation (32) and (33) with the help of the condition (37), we obtain the following solution

$$v(x, p) = \xi(x, p) + \{\zeta(p) - \xi(0, p)\} \exp\left[-x\left\{\frac{U}{2\lambda} - \left(\frac{U^2}{4\lambda^2} + A^2 + (n+1)(n+\alpha+\beta) + \frac{c'p}{\lambda}\right)^{1/2}\right\}\right] \qquad \ldots(41)$$

and

$$v'(x, p) = \Big[\{\zeta'(p) - \xi'(0, p)\}\, e^{-(Ux/2\lambda)} \sinh\left\{\left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right)^{1/2} (a - x)\right\}$$

$$+ \xi'(a, p)\, e^{-(Ux/2\lambda)} \cosh\left\{\left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right)^{1/2} x\right\}\Big] \Big/ \sinh\left\{\left(\frac{U^2}{4\lambda^2}\right.\right.$$

$$\left.\left. + B + \frac{c'p}{\lambda}\right)^{1/2} a\right\} + \tfrac{1}{2}\left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right)^{-(1/2)} \xi'(x, p) \qquad \ldots(42)$$

respectively, where

$$\xi(x, p) = \frac{1}{\lambda_1 - \lambda_2}\left[e^{\lambda_1 x} \int_0^x e^{-\lambda_1 x}\, t(x)\, dx - e^{\lambda_2 x} \int_0^x e^{-\lambda_2 x}\, t(x)\, dx\right] \qquad \ldots(43)$$

$$\xi'(x, p) = \frac{1}{\lambda_1' - \lambda_2'}\left[e^{\lambda_1' x} \int_0^x e^{-\lambda_1^1 x}\, t'(x)\, dx + e^{\lambda_2' x} \int_0^x e^{-\lambda_2' x}\, t^1(x)\, dx\right] \qquad \ldots(44)$$

where

$$\lambda_1 = \frac{U}{2\lambda} - \left[\frac{U^2}{4\lambda^2} + A^2 + (n+1)(n+\alpha+\beta) + \frac{c'p}{\lambda}\right]^{1/2}$$

$$\lambda_2 = \frac{U}{2\lambda} + \left[\frac{U^2}{4\lambda^2} + A^2 + (n+1)(n+\alpha+\beta) + \frac{c'p}{\lambda}\right]^{1/2}$$

$$\lambda_1' = \frac{U}{2\lambda} - \left[\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right]^{1/2}$$

$$\lambda_2' = \frac{U}{2\lambda} + \left[\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right]^{1/2}.$$

The value of $u(x, y, z, t)$ can be obtained by taking inverse Laplace of $v(x, p)$, evaluating the integral (23) after substituting the value of $u(x, t)$ thus obtained and finally taking inversion of the resulting quantity in the generalized Jacobi transform.

The value of $v(x, p)$ obtained in (41) is in complicated form. Here, we calculate it, and consequently $u(x, y, z, t)$ for a simple set of extreme and initial conditions.

Let

$$\left.\begin{aligned} &\theta(y, z, t) = f(z)\, e^{-p_1 t - p_2 y},\ Q(x, y, z, t) = 0 \\ &\eta(x, y, z) = 0,\ \phi(x, z, t) = 0 \end{aligned}\right\} \qquad \ldots(45)$$

then using well known result

$$L\{e^{-\alpha t}\} = \frac{1}{p + \alpha} \qquad \ldots(46)$$

and the inversion formula (30), we find that (41) gives

$$u_{ns}(x, t) = \frac{A\, e^{-(Ux/2\lambda)} f_n}{(\rho_2^2 + A^2)\sqrt{\pi}} \cdot \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\exp\left[x\left\{\frac{U^2}{4\lambda^2} + A^2 + (n+1)(n+\alpha+\beta) + \frac{c'p}{\lambda}\right\}^{1/2} + pt\right]}{p + \rho_1}\, dp. \qquad \ldots(47)$$

The integrand of (47) has simple pole at $p = -\rho_1$. Evaluating the residue at this pole, we arrive at the following value of $u(x, y, z, t)$:

$$u(x, y, z, t) = \frac{2}{\lambda} \sum_{n=0}^{\infty} \frac{(z-c)^{\beta}(d-z)^{\alpha}}{\delta_n} \times P_n^{(\alpha,\beta)}(z) \int_0^{\infty} \sin Ay\, u_{ns}(x, t)\, dA \qquad \ldots(48)$$

where

$$u_{ns}(x, t) = \frac{A f_n}{(\rho_2^2 + A^2)} \exp\left[-\frac{Ux}{2\lambda} + x\left\{\frac{U^2}{4\lambda^2} + A^2 + (n+1)(n+\alpha+\beta) - \frac{c'\rho_1}{\lambda}\right\}^{1/2} - \rho_1 t\right] \qquad \ldots(49)$$

and

$$\delta_n = \frac{(d-c)^{\alpha+\beta+1}\, \Gamma(\alpha+n+1)\, \Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\, \Gamma(\alpha+\beta+n+1)}. \qquad \ldots(50)$$

Similarly the value $u(x, y, z, t)$, in the case of the finite slab; can be obtained by proceeding on the same line. In this case, we assume

$$\left.\begin{aligned} \theta(y, z, t) &= f(y, z), \quad Q(x, y, z, t) = 0 \\ \eta(x, y, z) &= 0, \quad \phi(x, z, t) = 0 \end{aligned}\right\} \qquad \ldots(51)$$

Here, we have

$$v'(x, p) = \frac{f_{ns}\, e^{Ux/2\lambda} \sinh\left\{\left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right)^{1/2}(a - x)\right\}}{p \sinh\left\{\left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda}\right)^{1/2} a\right\}} \qquad \ldots(52)$$

where

$$f_{ns} = \int_0^b \int_c^d P_n^{(\alpha,\beta)}(z) f(y, z)\, dz \sin\frac{m\pi y}{b}\, dy. \qquad \ldots(53)$$

Using the inversion formula (30), we arrive at

$$u_{ns}(x, t) = \frac{\sqrt{2}\, f_{ns}\, e^{Ux/2\lambda}}{\pi^{1/2}}$$

$$\times \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt} \sinh \left\{ \left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda} \right) (a - x) \right\}}{\sinh \left\{ \left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda} \right)^{1/2} a \right\}} \frac{dp}{p}. \qquad \ldots(54)$$

The integrand of (54) has simple poles at $p = 0$ and at

$$a\left(\frac{U^2}{4\lambda^2} + B + \frac{c'p}{\lambda} \right)^{1/2} = n\pi i,\ n = 1, 2, \ldots$$

that is $\quad p = -\frac{U^2}{4c'\lambda} - \frac{B\lambda}{c'} - \frac{\lambda n^2 \pi^2}{a^2 c'},\ n = 1, 2, \ldots .$

Evaluating the residues at these poles, we get

$$u_{ns}(x, t) = \sqrt{\frac{2}{\pi}}\, f_{ns}\, e^{Ux/2\lambda} \frac{\sinh \left\{ \left(\frac{U^2}{4\lambda^2} + B \right)^{1/2} (a - x) \right\}}{\sinh \left\{ \left(\frac{U^2}{4\lambda} + B \right)^{1/2} a \right\}}$$

$$+ \frac{2^{3/2} \pi^{1/2} f_{ns}\, e^{Ux/2\lambda}}{a^2} \sum_{r=1}^{\infty} \frac{(-1)^r r \sin \left[\frac{r\pi(a - x)}{a} \right] e^{-\left[\frac{U^2}{4\lambda} + \frac{\lambda r^2 \pi^2}{a^2 c'} \right] b}}{\left[\frac{U^2}{4\lambda^2} + \frac{r^2 \pi^2}{a^2} \right]} \qquad \ldots(55)$$

Applying the inversion formulae (24) and (11), we obtain finally

$u(x, y, z, t) =$

$$\frac{4e^{Ux/2\lambda}}{\pi b} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(z - c)^{\beta} (d - z)^{\alpha}\, n!(\alpha + \beta + 2\lambda + 1)\, \Gamma(\alpha + \beta + n + 1)}{(d - c)^{\alpha+\beta+1}\, \Gamma(\alpha + n + 1)\, \Gamma(\beta + n + 1)}$$

$$\times \sin \frac{m\pi y}{b} \frac{\sinh \left\{ \left(\frac{U^2}{4\lambda^2} + B \right)^{1/2} (a - x) \right\}}{\sinh \left\{ \left(\frac{U^2}{4\lambda^2} + B \right)^{1/2} a \right\}} f_{ns} + \frac{4e^{Ux/2\lambda}}{\pi^{1/2} b}$$

$$\times \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^r r \sin \left[\frac{r\pi(a - x)}{a} \right] e^{-\left(\frac{U^2}{4\lambda} + \frac{\lambda r^2 \pi^2}{a^2 c'} \right) t} (z - c)^{\beta} (d - z)^{\alpha}\, n!\, (\alpha + \beta + 2n + 1)\, \Gamma(\alpha + \beta + n + 1)}{\left[\frac{U^2}{4\lambda^2} + \frac{r^2 \pi^2}{a^2} \right] \Gamma(a + n + 1)\, \Gamma(\beta + n + 1)\, (d - c)^{\alpha+\beta+1}}$$

$$\times P_n^{(\alpha,\beta)}(z) \sin \frac{m\pi y}{b} f_{ns}.$$

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