A NOTE ON A PAPER BY KRA

by RANJAN ROY, *Department of Mathematics, University of Kentucky, Lexington, Kentucky* 40506

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The more general definition of a deformation given below is required to include the branched structures considered by Mandelbaum (1972). Some results of Kra (1972) are generalized in this paper so that they hold for the new definition of deformation.

§1. We (Roy 1974) gave the following definition of a deformation of a Fuchsian group. Let G be a Fuchsian group acting on the upper half plane U. A deformation of G is a pair (χ, f) where (i) χ is a homomorphism of G into the group of Mobius transformations, and (ii) f is a local homeomorphism except at a finite number of points z_i $(i = 1, \ldots, n)$ in a fundamental domain of G, and f is meromorphic on U such that

$$f \circ A = \chi(A) \circ f \text{ for } A \in G. \qquad \ldots(1)$$

This definition is a generalization of the one given by Kra (1972) and is motivated by the work of Mandelbaum (1972). In this note we shall generalize some results of Kra (1972).

Suppose f is a function meromorphic on a domain D, then the Schwartzian derivative Sf is defined on D by the formula

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \qquad \ldots(2)$$

If h is a function meromorphic on $f(D)$ then

$$S(h \circ f) = S(h) \circ f\, f'^2 + Sf. \qquad \ldots(3)$$

Also $Sf = 0$ if and only if f is a Mobius transformation.

Let $D = \sum_{i=1}^{n} (m_i - 1)\, z_i$ be a divisor on the fundamental domain of G, that is, the points z_i belong to the fundamental domain and m_i are integers $\geqslant 1$ and the summation is a formal sum.

A deformation (χ, f) is of type $D = \sum_{i=1}^{n} (m_i - 1) z_i$ if f is an m_i to 1 map at z_i $(i = 1, ..., n)$ and a local homeomorphism at all other points of the fundamental domain.

We call a deformation special if, whenever $A \in G$ is parabolic, $\chi(A)$ is either parabolic, the identity, or an elliptic element of finite order, and $Sf(z)$ has finite limits as z approaches the parabolic fixed points through cusped regions belonging to the fixed points.

Let $p_1, ..., p_s$ be a complete set of inequivalent fixed points of parabolic elements $A_1, ..., A_s$ (respectively) of G. Let A_i generate the stability subgroups G_{p_i}. Suppose B_i $(i = 1, ..., s)$ are Mobius transformations which fix U and have the following properties

(i) $B_i(\infty) = p_i$

(ii) $B_i^{-1} \circ A_i \circ B_i(z) = z + 1$

(iii) if $U_c = \{z \in C; \operatorname{Im} z > c\}$

then for sufficiently large c two points of $B(U_c)$ are equivalent under G if and only if they are equivalent under the cyclic group generated by A_i. These B_i exist as shown by Ahlfors (1964). We then say that the deformation (χ, f) has the limit α_i at p_i if

$$\lim_{z \to i\infty} S(f \circ B_i) = 2\pi^2 \alpha_i^2 .$$

It is easy to show that this limit is independent of B_i.

With the above notations suppose (χ, f) is a special deformation such that χ takes $A_1, ..., A_j$ to either parabolic elements or the identity and $A_{j+1}, ..., A_s$ to elliptic elements of order $d_{j+1}, ..., d_s$ respectively.

We shall also need the following lemma given by Roy (1974).

Lemma — Let (χ, f) be a special deformation of type D. Then (χ, f) has limits k_i $(i = 1, ..., j)$ and $\frac{l_i}{d_i}$ $(i = j + 1, ..., s)$ where k_i and l_i are integers. Further if (a) the limit of (χ, f) at p_i is zero then $\chi(A_i)$ is parabolic and f and f' have limits as $z \to p_i$. The limit of f' is finite and non-zero.

(b) If the limit at p_i is $\frac{l_i}{d_i}$ (where d_i may be one) and $\tilde{f} = B_i^{-1} \circ f \circ B_i$, then

$$\tilde{f}'(z) = \sum_{\pm l_i}^{\infty} a_n e^{2\pi i n z / d_i}, \; z \in U \qquad \ldots(4)$$

and $$\tilde{f}(z) = \sum_{\pm l_i}^{\infty} b_n e^{2\pi i n z/d_i} + bz, \ z \in U. \qquad \ldots(5)$$

where $b \neq 0$ if $\chi(A_i)$ is parabolic.

Now suppose D is any domain in $CU\{\infty\}$ whose universal covering space is U, the upper half plane. Let $\pi : U \to D$ be a covering map. The Poincare metric for D, $\lambda_D(z) \mid dz \mid$, is defined by settings

$$\lambda_U(z) = (\text{Im } z)^{-1}, \ z \in U \text{ and} \qquad \ldots(6)$$

$$\lambda_{\pi(U)}(\pi(z)) \mid \pi'(z) \mid = \lambda_U(z), \ z \in U. \qquad \ldots(7)$$

It is easy to check that λ_D is well defined, and that $\lambda_D(z) \mid dz \mid$ is a conformal invariant, that is, $\lambda_{f(D)}(f(z)) \mid f'(z) \mid = \lambda_D(z)$ for all conformal maps f of D. If ϕ is holomorphic we define

$$\| \phi \|_D = \sup \{\lambda_D(z)^{-2} \mid \phi(z) \mid ; z \in D\} < \infty.$$

We also need the following definition. A function ϕ on U is said to be a multiplicative q-form for a Fuchsian group G if

$$\phi(Az)\, A'(z)^q = C_A \phi(z), \ \ z \in U, \ \ A \in B \qquad \ldots(8)$$

where C_A depends only on A. If $C_A = 1$ for all $A \in G$ then we have a q-form.

§2. For a group of finite type G (that is, G is a finitely generated Fuchsian group of the first kind) acting on U, we define

$A(G) = 2g - 2 + \sum_{i=1}^{p} (1 - 1/l_i) + q$, where g is the genus of the Riemann surface U/G, q is the number of punctures of U/G and p is the number of ramified points $l_i (i = 1, \ldots, p)$.

If (χ, f) is a special deformation of G of type $D = \sum_{i=1}^{n} (m_i - 1) z_i$ and with limits $\frac{k_i}{t_i}$ $(i = 1, \ldots, q)$ then put

$$O(f) = \sum_{i=1}^{n} \frac{m_i - 1}{l_i} + \sum_{i=1}^{q} \pm k_i, \text{ where } l_i \text{ is the order of the group } G_{z_i},$$

and the sign in $\pm k_i$ is determined by the expansion of $\tilde{f}$ in (5).

Theorem — (a) If (χ, f) is a special deformation of a group G of finite type, and (χ, f) is of type D then $f(U)$ is not conformally equivalet to C, provided $O(f) \neq A(G)$.

(b) Let G be a finite type and (χ, f) be a deformation of G, of type D and with limit zero at each of the fixed points. Then the fundamental group of $f(U)$ is not infinite cyclic provided $O(f) \neq A(G)$.

PROOF : (a) If $f(U)$ is conformal to C, there exists a conformal map B from $f(U)$ onto C. Let $f_1 = B \circ f$ and $\chi_1(A) = B \circ \chi(A) \circ B^{-1}$ so that we can work with (χ_1, f_1) instead of (χ, f). Note that $f_1(U) = C$. From (1) and the fact that the only conformal self map of C is of the form $az + b$ it follows that

$$f_1(Az) = a_A f_1(z) + b_A \text{ for } A \in G.$$

Hence $f_1'(Az)\, A'(z) = a_A f_1'(z)$, and f_1 is a multiplicative 1-form. According to Lehner (1964), such a function has degree (sum of orders of zeros — sum of orders of poles) equal to $A(G)$. Using the lemma it can be seen easily that the degree is also $O(f)$.

If $O(f) \neq A(G)$ it follows that $f_1' = 0$, that is, $f =$ constant which is a contradiction.

(b) Every region whose fundamental group is infinite cyclic is conformal to

$$E = \{ z \in C;\ a < |z| < b,\ 0 \leqslant a < b \leqslant \infty \}.$$

Hence there exists a conformal map g from $f(U)$ to such a region. Consider (χ_1, f_1) where $f_1 = g \circ f$ and $\chi_1(A) = g \circ \chi(A) \circ g^{-1}$. $\chi_1(A)$ is a conformal map of E, hence it is either of the form

$$z \to kz \quad \text{or} \quad z \to k/z,\ k \in C - \{0\}. \qquad \ldots(9)$$

These are Mobius transformations and hence (χ_1, f_1) is also a deformation of G. We shall now show that (χ_1, f_1) also has limit zero at each of the parabolic fixed points. We may assume that the fixed point is ∞ and the parabolic element is $Az = z + 1$. By Ahlfor's lemma mentioned in section 1, there is a region $U_c = \{z \in C;\ \text{Im } z > c\}$ such that f_1 is univalent in it. From (1), (2) and (3) it follows that

$$Sf(Bz)\, B'(z)^2 = Sf(z) \text{ for } B \in G,$$

so that for $Az = z + 1$, $Sf(z + 1) = Sf(z)$.

Thus Sf has a Fourier series expansion and from the fact that $Sf \to 0$ at the fixed point we get

$$Sf(z) = \sum_{1}^{\infty} a_n e^{2\pi i n z}.$$

Now $\lambda_{U_c}(z) = \dfrac{1}{y - c}$ where $y = \text{Im } z$. This follows from (6) and (7).

Hence $\quad \| Sf \|_{U_c} = \sup \{\lambda_{U_c}(z)^{-2} \mid Sf(2) \mid ; z \in U_c\} < M.$

By using Lemma 2 of Kra (1972) we get

$$\| S(g \circ f) \|_{U_c} < M_1.$$

This implies that $S(g \circ f) \to 0$ as $z \to$ fixed point and (χ_1, f_1) is also a special deformation with limit zero at the fixed points. We have therefore shown that we can assume $f(U) = E$. Now since neither of the transformations (9) is parabolic it follows from part (a) of the lemma in section 1 that G cannot have parabolic elements if $f(U) = E$, that is, if G has parabolic elements $f(U)$ is not conformal to E. Now suppose there are no parabolic elements in G. Then the function

$$h(z) = \left(\frac{f'(z)}{f(z)}\right)^2$$

can be shown to satisfy the relation

$$h(Az)\, A'(z)^2 = h(z) \text{ for } z \in U,\ A \in G, \text{ provided } f(U) = E.$$

Thus h is a 2-form and its degree is $2A(G)$. However the degree of h is also $2\ O(f)$. Hence $O(f) \neq A(G)$ then $f(U)$ is not conformal to E, that is the fundamental group of $f(U)$ is not infinite cyclic.

Remark : $\chi(G)$ is a group of conformal self-mappings of $f(U)$. Thus $\chi(G)$ is discontinuous unless $f(U)$ is conformally equivalent to the sphere, the unit disc, the complex plane, the punctured plane, the punctured disc or the annulus (see Springer 1957). The last four possibilities are ruled out if $O(f) \neq A(G)$ and (χ, f) is a special deformation with zero limits. Thus under these conditions on (χ, f) if $f(U)$ is not the sphere or a simply connected region with more than 2 boundary points, then $\chi(G)$ is discontinuous.

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