MAGIC PARTITIONS

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Let $S = \{(a_1, a_2, a_3, a_4) : a_1 + a_2 + a_3 + a_4 = r\}$ where a's are non-negative integers, be a set of 4-partite numbers. In this paper, formulae are obtained for the numbers of partitions of the 4-partite number (r, r, r, r) into the elements of S, both when the summands in a partition can be repeated and also when they are required to be distinct. Why such partitions are called "Magic" is obvious. It is note-worthy that the pattern of results agrees with that in the case of partitions of j-partite numbers in general.

1. INTRODUCTION

In what follows, small letters other than x, denote non-negative integers, unless stated otherwise.

Let S denote the set of j-partite numbers defined by

$$S = S(r) = \{(a_1, a_2, \ldots, a_j) : a_1 + a_2 + \ldots + a_j = r\}. \qquad \ldots(1)$$

Then the problem is to find the number of partitions of the j-partite number $(r, r, \ldots, r)$ into the elements of S. The number of partitions will be denoted by $u(r)$ when the parts can be repeated and by $q(r)$ when they are required to be distinct. We write

$$U(r) = j!\, u(r), \quad Q(r) = j!\, q(r). \qquad \ldots(2)$$

Why we call such partitions 'magic partitions' is obvious.

The attention of the reader is invited to author's earlier paper (Gupta 1961) wherein he had obtained formulae for the numbers of partitions of j-partite numbers into j-partite numbers, both when parts in a partition could be repeated and also when they were required to be distinct. The argument used there goes through completely in the present case also but what form the P_0 and Q_0 of the earlier paper will have now is difficult to say.

In this paper, we consider the case $j = 4$ in detail. The results for $j = 2, 3$ are easier to obtain and will be just stated.

2. BASIC PROBLEMS

Let us first find the number $w(r, t)$ of solutions of the equation

$$x_1 + x_2 + \ldots + x_j = r \qquad \ldots(3)$$

in non-negative integers in which no $x \geqslant t$.

For the restriction on the values of x's to be meaningful, it is evident that t cannot be less than r/j.

Let

$$a_1 + a_2 + \dots + a_j = r$$

be any solution of (3), not necessarily satisfying the given condition.

Let

$$a_i = b_i + t\, h_i, \quad 0 \leqslant b_i < t, \quad i = 1, 2, \dots, j. \qquad \dots(4)$$

If

$$h_1 + h_2 + \dots + h_j = m, \quad 0 \leqslant m \leqslant r/t \qquad \dots(5)$$

then we have

$$b_1 + b_2 + \dots + b_j = r - mt. \qquad \dots(6)$$

Since each a_i determines and is uniquely determined by h_i and b_i, the total number of solutions of (3) in non-negative integers, must be the same as the number of solutions of the simultaneous equations (5) and (6).

Hence, we must have

$$\sum_{m=0}^{[r/t]} w(r - mt, t) \binom{m + j - 1}{j - 1} = \binom{r + j - 1}{j - 1}. \qquad \dots(7)$$

Writing k for $[r/t]$, (7) yields the useful result

$$w(r, t) = \sum_{m=0}^{k} (-1)^m \binom{j}{m} \binom{r - tm + j - 1}{j - 1}. \qquad \dots(8)$$

For $t = [r/2] + 1$, we thus have

$$w(r, t) = \binom{r + j - 1}{j - 1} - j \binom{r - [r/2] + j - 2}{j - 1}. \qquad \dots(9)$$

Let us next answer the following question :

How many of the solutions of the equation

$$x_1 + x_2 + \dots + x_j = r$$

in non-negative integers, have

$$x_1 > t_1 \text{ and } x_2 > t_2\ ?$$

For the problem to be non-trivial, we must obviously have

$$t_1 + t_2 + 2 \leqslant r.$$

Let us first find the number of those solutions which satisfy only the condition on x_1.

Suppose

$$x_1 = t_1 + s_1, \text{ where } 1 \leqslant s_1 \leqslant r - t_1.$$

Then the required number of solutions is

$$\sum_{s=1}^{r-t_1} \binom{r - t_1 - s_1 + j - 2}{j - 2} = \binom{r - t_1 + j - 2}{j - 1}.$$

We can now consider the case of solutions satisfying the two conditions together.

Take

$$x_1 = t_1 + s_1, \quad x_2 = t_2 + s_2 \qquad \text{...(10)}$$

with

$$1 \leqslant s_1 \leqslant r - t_1 - t_2 - 1, \quad 1 \leqslant s_2 \leqslant r - t_1 - t_2 - s_1.$$

Then the number of solutions which satisfy the two conditions simultaneously, is given by

$$\sum_{s_1=1}^{r-t_1-t_2-1} \quad \sum_{s_2=1}^{r-t_1-t_2-s_1} \binom{r - t_1 - s_1 - t_2 - s_2 + j - 3}{i - 3}$$

$$= \sum_{s_1=1}^{r-t_1-t_2-1} \binom{r - t_1 - t_2 - s_1 + j - 3}{j - 2}$$

$$= \binom{r - t_1 - t_2 + j - 3}{- 1}. \qquad \text{...(11)}$$

As a direct consequence of the above results, we can now state that

The number of solutions of

$$x_1 + x_2 + \ldots + x_j = r \qquad \text{...(12)}$$

in non-negative integers, which have

$$x_1 \leqslant t_1, \quad x_2 \leqslant t_2$$

simultaneously, is given by

$$\binom{r+j-1}{j-1} - \binom{r-t_1+j-2}{j-1} - \binom{r-t_2+j-2}{j-1} + \binom{r-t_1-t_2+j-3}{j-1}.$$

This result admits of extension in the usual manner.

Finally, we ask :

If all the solutions of (12) were to be written out, how often will a particular integer $t \leqslant r$, have to be written ?

Since the number of solutions which have $x_1 = t$ is

$$\binom{r-t+j-2}{j-2}$$

the required answer is immediately seen to be

$$j\binom{r-t+j-2}{j-2}. \qquad ...(13)$$

3. Solution of the Problem

We are now adequately armed to attack our problem.

Denote by v_4 the number of partitions of (r, r, r, r) into four equal elements of S with $j = 4$; by v_{31} the number of those which have three equal parts and the fourth distinct from them; by v_{22} the number of those which have two distinct pairs of equal parts; and by v_{211} the number of the partitions which have two parts equal and two others distinct from them and also from each other. Then, we readily see that

$$v_4 = 1 \text{ or } 0 \qquad ...(14)$$

according as r is or is not a multiple of 4.

The generating function of v_4, therefore, is

$$1/(1-x^4). \qquad ...(15)$$

The number of partitions which have at least three parts equal, is given by

$$v_{31} + v_4 = \binom{r+3}{3} - 4\binom{r-[r/3]+2}{3} + 6\binom{r-2[r/3]+1}{3}. \qquad ...(16)$$

The generating function of $v_{31} + v_4$, is

$$(1+x^4+x^8)/(1-x^3)^4. \qquad ...(17)$$

Again, the number of partitions in which two parts are equal and the remaining two parts also equal but not necessarily distinct from the first two, is $2\nu_{22} + \nu_4$. It is zero when r is odd.

When r is even, with $h = r/2$, we have

$$2\nu_{22} + \nu_4 = \binom{r+3}{3} - 4\binom{h+2}{3} = 4\binom{h+2}{3} + \binom{h+1}{1}. \qquad \ldots(18)$$

The corresponding generating function is

$$(1 + x^2)^2/(1 - x^2)^4. \qquad \ldots(19)$$

We consider the case of ν_{211} in the next section.

4. Evaluation of ν_{211}

The number of partitions in which at least two parts are equal, is given by

$$\nu_{211} + 2\nu_{22} + \nu_{31} + \nu_4. \qquad \ldots(20)$$

Let (a_1, a_2, a_3, a_4) represent the part which appears twice in the partition. Then, we are left to partition

$$(r - 2a_1,\ r - 2a_2,\ r - 2a_3,\ r - 2a_4) \qquad \ldots(21)$$

into two parts.

Since the part (a_1, a_2, a_3, a_4) occurs twice in the partition,

$$a_i \leqslant r/2,\ \ i = 1, 2, 3, 4. \qquad \ldots(22)$$

Also

$$a_1 + a_2 + a_3 + a_4 = r. \qquad \ldots(23)$$

Hence, each of the elements of the 4-partite number in (21) is $\leqslant r$, while the sum of the four elements together is $2r$. If one part in the partition of (21) into two parts is

$$(c_1, c_2, c_3, c_4) \text{ with } c_1 + c_2 + c_3 + c_4 = r; \qquad \ldots(24)$$

then we must have

$$c_i \leqslant r - 2a_i,\ i = 1, 2, 3, 4. \qquad \ldots(25)$$

What we do now is counting the solutions of (24) which do not satisfy at least one of the conditions in (25).

Making use of the results of section 2 and letting a_1, a_2, a_3, a_4 run over all the solutions of (23) satisfying conditions in (22), we get

$$v_{211} + 2v_{22} + v_{31} + v_4$$

$$= \left\{1 + \binom{r+3}{3}\right\} \left\{\binom{r+3}{3} - 4\binom{h+2}{3}\right\}$$

$$- 4 \sum_{m=1}^{h} \left\{\binom{r-m+2}{2} - 3\binom{h-m+1}{2}\right\} \binom{2m+2}{3}$$

$$+ 6 \sum_{m=1}^{h} m^2 \binom{r-2m+3}{3}; \qquad \ldots(26)$$

or

$$\binom{r+3}{3} \left\{\binom{r+3}{3} - 4\binom{h+3}{3}\right\}$$

$$- 4 \sum_{m=0}^{h} \left\{\binom{r-m+2}{2} - 3\binom{h-m+2}{2}\right\} \binom{2m+2}{3}$$

$$+ 12 \sum_{m=0}^{h} \binom{m+2}{2} \binom{r-2m-1}{3} \qquad \ldots(27)$$

according as r is even or odd with $h = [r/2]$ in each case. The sigmas are not too difficult to evaluate and we get

$$v_{211} + 2v_{22} + v_{31} + v_4$$

$$= 64\binom{h+5}{6} - 96\binom{h+4}{5} + 46\binom{h+3}{4} - 5\binom{h+2}{3} + \binom{h+2}{2}$$

or

$$64\binom{h+6}{6} - 128\binom{h+5}{5} + 80\binom{h+4}{4} - 16\binom{h+3}{3};$$

according as r is even or odd.

Using (18), this yields

$$2v_{211} + 2v_{22} + 2v_{31} + v_4$$

$$= 128\binom{h+5}{6} - 192\binom{h+4}{5} + 92\binom{h+3}{4} - 14\binom{h+2}{3}$$

$$+ 2\binom{h+2}{2} - \binom{h+1}{1}$$

or

$$128\binom{h+6}{6} - 256\binom{h+5}{5} + 160\binom{h+4}{4} - 32\binom{h+3}{3} \quad ...(28)$$

according as r is even or odd.

The generating function corresponding to this on simplification turns out to be

$$(1 + 11x^2 + 32x^3 + 52x^4 + 64x^5 + 52x^6 + 32x^7 + 11x^8 + x^{10})/(1 - x^2)^7. \quad ...(29)$$

5. Formulae for $U(r)$ and $Q(r)$

We now observe that the total number of 4×4 magic matrices having each line sum equal to r is known to be (Anand *et al.* 1966)

$$\binom{r+3}{3} + 20\binom{r+4}{5} + 152\binom{r+5}{7} + 352\binom{r+6}{9}. \quad ...(30)$$

This must be the same as

$$Q(r) + 12v_{211} + 6v_{22} + 4v_{31} + v_4. \quad ...(31)$$

Moreover,

$$U(r) = Q(r) + 24\,(v_{211} + v_{22} + v_{31} + v_4). \quad ...(32)$$

Relations (14, 16, 18, 28, 30, 31) determine $Q(r)$ and these along with (32) then suffice for the evaluation of $U(r)$ in terms of combinatory functions. The resulting expression is in each case a polynomial in r of degree nine.

The generating function of $Q(r)$ is easily found to be

$$(1 + 14x + 87x^2 + 148x^3 + 87x^4 + 14x^5 + x^6)/(1 - x)^{10}$$
$$- 6(1+11x^2 + 32x^3 + 52x^4 + 64x^5 + 52x^6 + 32x^7 + 11x^8 + x^{10})/(1 - x^2)^7$$
$$+ 3(1 + x^2)^2/(1 - x^2)^4 + 8(1 + x^4 + x^8)/(1 - x^3)^4 - 6/(1 - x^4) \quad ...(33)$$

As could be expected, the generating function of $U(r)$ is obtained by taking the second and the fifth terms in (33) with a plus sign. It is note-worthy that the numerator of each term is a reciprocal polynomial. The first term provides the generating function of (30).

The corresponding results for $j = 2, 3$ are

$$1/(1 - x)^2 - 1/(1 - x^2)$$

and

$$(1 - x^3)/(1 - x)^6 - 3(1 + x^3)/(1 - x^2)^3 + 2/(1 - x^3)\,;$$

respectively and our remark regarding the generating function of $U(r)$ still holds good as the function is obtained by changing the sign of the second term.

For $j = 4$, a short table of values of v_4, v_{31}, v_{22}, v_{211}, $q(r)$ and $u(r)$ is appended (Table I).

TABLE I

r	v_4	v_{31}	v_{22}	v_{211}	q	u
0	1	0	0	0	0	1
1	0	0	0	0	1	1
2	0	0	3	6	8	17
3	0	4	0	12	77	93
4	1	0	9	69	386	465
5	0	0	0	144	1602	1746
6	0	10	22	372	5337	5741
7	0	4	0	684	15550	16238
8	1	0	42	1440	40167	41650
9	0	20	0	2332	95055	97407
10	0	10	73	4254	2 08075	2 12412
11	0	4	0	6492	4 28271	4 34767
12	1	34	115	10655	8 34561	8 45366
13	0	20	0	15436	15 53888	15 69344
14	0	10	172	23706	27 77808	28 01696

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