CERTAIN THEOREMS IN TWO DIMENSIONAL LAPLACE TRANSFORM

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In the present paper we have established three theorems in two-dimensional Laplace transform and have evaluated many double integrals by the application of these theorems.

§1. Laplace (1812), considered the integral equation

$$F(p) = p \int_0^\infty e^{-px} f(x)\, dx, \quad \text{Re}\,(p) > 0. \qquad \ldots(1.1)$$

Humbert (1934) gave the generalization of (1.1) as

$$F(p, q) = pq \int_0^\infty \int_0^\infty e^{-(px+qy)} f(x, y)\, dx\, dy, \; \text{Re}\,(p, q) > 0. \qquad \ldots(1.2)$$

Symbolically (1.1) and (1.2) have been denoted respectively as

$$F(p) \doteqdot f(x) \quad \text{and} \quad F(p, q) \doteqdot f(x, y).$$

Ditkin and Prudnikov (1962, pp. 51-68) have considered many correspondences in two-dimensional Laplace transform (1.2) established from the known one's in (1.1). In the present paper further we have established results in (1.2) by applying Goldstein's theorem and using some known results in (1.1). Results established here are of course of different nature than those established in Ditkin and Prudnikov (1962). These results are useful in evaluating double integrals involving parabolic cylinder functions. As illustrations we have evaluated a few integrals by the application of these results.

§2. *Theorem* 1 — If

$$f_1(x, y) \doteqdot \phi_1(p, q) \qquad \ldots(2.1)$$

$$x^{2\rho} y^{2\mu} f_1(x^2, y^2) \doteqdot \theta_1(p, q) \qquad \ldots(2.2)$$

then

$$\theta_1(\alpha, \beta) = \frac{\alpha\beta}{2^{(2+\mu+\rho)}\pi} \int_0^\infty \int_0^\infty \phi_1(x, y)\, x^{-\rho-3/2}\, y^{-\mu-3/2}\, e^{-\frac{1}{8}\left(\frac{\alpha^2}{x}+\frac{\beta^2}{y}\right)}$$

$$\times D_{2\rho}\left(\frac{\alpha}{\sqrt{(2x)}}\right) D_{2\mu}\left(\frac{\beta}{\sqrt{(2y)}}\right) dx\, dy \qquad \ldots(2.3)$$

provided $f_1(x, y)$ and $x^{\rho-1/2}\, y^{\mu-1/2}\, f_1(x, y)$ are absolutely integrable in $0 \leqslant x < \infty$, $0 \leqslant y < \infty$, $\text{Re}\,(\alpha, \beta) > 0$.

PROOF : We know from Erdélyi (1954, p. 246) that

$$x^{-\rho-1/2}\, e^{-(\alpha/8x)}\, D_{2\rho}\, (\sqrt{(\alpha/2x)}) \doteqdot \sqrt{\pi}\, p^{\rho+1/2}\, 2^{\rho}\, e^{-\sqrt{(\alpha p)}},\ \text{Re}\,(\alpha) > 0$$

from which we deduce

$$2^{-\mu-\rho}\, x^{-\rho-1/2}\, y^{-\mu-1/2}\, e^{-\alpha/8x-\beta/8y}\, D_{2\rho}\, (\sqrt{(\alpha/2x)})\, D_{2\mu}\, (\sqrt{(\beta/2y)})$$

$$\doteqdot \pi\, p^{\rho+1/2}\, q^{\mu+1/2}\, e^{-\sqrt{(\alpha p)}-\sqrt{(\beta q)}},\ \text{Re}\,(\alpha, \beta) > 0. \qquad \ldots(2.4)$$

Now applying the well-known Goldstein's theorem to (2.1) and (2.4), we get

$$\int_0^\infty \int_0^\infty f_1(x, y)\, x^{\rho-1/2}\, y^{\mu-1/2}\, e^{-\sqrt{(\alpha x)}-\sqrt{(\beta y)}}\, dx\, dy$$

$$= \frac{1}{2^{(\mu+\rho)}\pi} \int_0^\infty \int_0^\infty \phi_1(x, y)\, x^{-\rho-3/2}\, y^{-\mu-3/2}\, e^{-\alpha/8x-\beta/8y}\, D_{2\rho}(\sqrt{(\alpha/2x)})\, D_{2\mu}(\sqrt{(\beta/2y)})\, dx\, dy.$$

Now replacing x by x^2, y by y^2 on the left-hand side and α by α^2, β by β^2 on both the sides of the above equation and then using (2.2), we obtain the desired result.

Example 1 — Let $f_1(x, y) = x^{c_1}\, y^{c_2}\, G^{rs}_{uv}\left(cxy \,\middle|\, \begin{matrix} a_1, \ldots, a_u \\ b_1, \ldots, b_v \end{matrix}\right)$

$$\phi_1(p, q) = p^{-c_1}\, q^{-c_2}\, G^{r,s+2}_{u+2,v}\left(\frac{c}{pq} \,\middle|\, \begin{matrix} -c_1, -c_2, a_1, \ldots, a_u \\ b_1, \ldots, b_v \end{matrix}\right) \text{ (Jain 1970)}$$

provided $(u + v) < 2(r + s)$, $|\arg c| < \left(r + s - \frac{u}{2} - \frac{v}{2}\right)\pi$

and $\text{Re}\,(c_k + b_j + 1) > 0$, $k = 1, 2,\ j = 1, 2, \ldots, r$.

$$\theta_1(p, q) = \pi^{-1}\, p^{-2(c_1+\rho)}\, q^{-2(c_2+\mu)}\, 2^{2(c_1+c_2+\mu+\rho)}$$

$$\times G^{r,s+4}_{u+4,v}\left(\frac{2^4 c}{p^2 q^2} \,\middle|\, \begin{matrix} 0, \frac{1}{2} - c_1 - \rho,\ 0, \frac{1}{2} - c_2 - \mu,\ a_1, \ldots, a_u \\ b_1, \ldots\ldots\ldots\ldots\ldots\ldots, b_v \end{matrix}\right),$$

provided $u + v < 2(r + s)$, $|\arg c| < \left(r + s - \frac{u}{2} - \frac{v}{2}\right)\pi$,

$$\text{Re}\,(c_1 + b_j + 1/2 + \rho) > 0,$$

$$\text{Re}\,(c_2 + b_j + 1/2 + \mu) > 0,\ j = 1, 2, \ldots, r.$$

$\frac{0, 1}{2} - c_1 - \rho$ stands for $\frac{0}{2} - c_1 - \rho,\ \frac{1}{2} - c_1 - \rho$.

$$\therefore \int_0^\infty \int_0^\infty x^{-\left(c_1 + \rho + \frac{3}{2}\right)} y^{-\left(c_2 + \mu + \frac{3}{2}\right)} e^{-\frac{1}{8}\left(\frac{\alpha^2}{x} + \frac{\beta^2}{y}\right)} D_{2\rho}\left(\frac{\alpha}{\sqrt{(2x)}}\right)$$

$$\times D_{2\mu}\left(\frac{\beta}{\sqrt{(2y)}}\right) G^{r,s+2}_{u+2,v}\left(\frac{c}{xy}\middle|\begin{matrix} -c_1, -c_2, a_1, \ldots, a_u \\ b_1, \ldots\ldots\ldots\ldots\ldots, b_v \end{matrix}\right) dx\, dy$$

$$= \alpha^{-(2c_1+2\rho+1)} \beta^{-(2c_2+2\mu+1)} 2^{(2c_1+2c_2+3\mu+3\rho+2)}$$

$$\times G^{r,s+4}_{u+4,v}\left(\frac{2^4 c}{\alpha^2\beta^2}\middle|\begin{matrix} \frac{0, 1}{2} - c_1 - \rho, \frac{0, 1}{2} - c_2 - \mu, a_1, \ldots, a_u \\ b_1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, b_v \end{matrix}\right),$$

provided $u + v < 2(r + s)$, $|\arg c| < \left(r + s - \frac{u}{2} - \frac{v}{2}\right)\pi$, $\text{Re}(c_1 + b_j + \frac{1}{2} + \rho) > 0$, $\text{Re}\,(c_2 + b_j + \frac{1}{2} + \mu) > 0$, $\text{Re}\,(c_k + b_j + 1) > 0$, $j = 1, 2, \ldots, r, k = 1, 2$ and $\text{Re}\,(\alpha^2, \beta^2) > 0$.

Example 2 — Let $f_1(x, y) = \frac{1}{\pi\sqrt{(xy)}}$ be $r\{2(xy)^{1/4}\}$ and let $\mu = \rho = \frac{1}{2}$ in the theorem, then

$$\phi_1(p, q) = \sqrt{(pq)}\, J_0\left(\frac{1}{2\sqrt{(pq)}}\right) \text{ (Ditkin and Prudnikov 1962, p. 151)}$$

$$\theta_1(p, q) = \frac{p^2q^2}{\pi(p^2q^2 + 1)} \text{ (Ditkin and Prudnikov 1962, p. 135)}$$

$$\therefore \int_0^\infty \int_0^\infty (xy)^{-3/2} J_0\left(\frac{1}{2\sqrt{(xy)}}\right) e^{-\frac{1}{8}\left(\frac{\alpha^2}{x} + \frac{\beta^2}{y}\right)} D_1\left(\frac{\alpha}{\sqrt{(2x)}}\right)$$

$$\times D_1\left(\frac{\beta}{\sqrt{(2y)}}\right) dx\, dy = \frac{8\alpha\beta}{1 + \alpha^2\beta^2},\ \text{Re}\,(\alpha^2, \beta^2) > 0.$$

Example 3 — Let $f_1(x, y) = \begin{cases} \frac{x^a}{\Gamma(a + 1)}, & y > x \\ \frac{y^a}{\Gamma(a + 1)}, & y < x \end{cases}$

and let $\mu = \rho = \frac{1}{2}$ in the theorem, then we get

$$\phi_1(p, q) = (p + q)^{-a}, \quad a > -1, \quad \text{(Ditkin and Prudnikov 1962, p. 136)}$$

$$\theta_1(p, q) = \frac{\sqrt{\pi}(a+2)\,\Gamma(a+3/2)}{p^{(2a+1)}\,q}\, {}_2F_1\left(a,\ a+3/2;\ 2a+3;\ \frac{p^2-q^2}{p^2}\right).$$

$$\therefore \int_0^\infty \int_0^\infty \frac{(xy)^{-2}}{(x+y)^a}\, e^{-\frac{1}{8}\left(\frac{\alpha^2}{x}+\frac{\beta^2}{y}\right)} D_1\left(\frac{\alpha}{\sqrt{(2x)}}\right) D_1\left(\frac{\beta}{\sqrt{(2y)}}\right) dx\,dy$$

$$= \frac{8(a+2)\,\pi^{3/2}\Gamma(a+3/2)}{\alpha^{(2a+2)}\,\beta^2}\, {}_2F_1\left(a,\ a+3/2;\ 2a+3;\ \frac{\alpha^2-\beta^2}{\alpha^2}\right),$$

$$a > -1, \quad \text{Re}\,(\alpha^2, \beta^2) > 0.$$

§3. *Theorem* 2 — Let $f_1(x, y) \doteqdot \phi_1(p, q)$ ...(3.1)

$$x^{(2\rho-3)}\, y^{(2\mu-3)}\, \phi_1\left(\frac{x^2}{4}, \frac{y^2}{4}\right) \doteqdot \theta_1(p, q) \qquad \text{...(3.2)}$$

then

$$\theta_1(\alpha, \beta) = \frac{\alpha\beta\Gamma(2\mu)\,\Gamma(2\rho)}{2^{(4-\mu-\rho)}} \int_0^\infty \int_0^\infty f_1(x, y)\, x^{-\rho}\, y^{-\mu}\, D_{-2\rho}\,(\alpha\sqrt{(2/x)})$$

$$\times\, D_{-2\mu}\left(\beta\sqrt{\left(\frac{2}{y}\right)}\right) e^{\frac{1}{2}\left(\frac{\alpha^2}{x}+\frac{\beta^2}{y}\right)} dx\,dy, \quad \text{Re}\,(\alpha, \beta) > 0, \ \text{Re}\,(\mu, \rho) > 0.$$

...(3.3)

PROOF : By using the known result (Erdélyi 1954, p. 147)

$$(2x)^{\rho-1}\, e^{-2\sqrt{(\alpha x)}} \doteqdot \Gamma(2\rho)\, p^{-\rho+1}\, e^{\alpha/2p}\, D_{-2\rho}\,(\sqrt{(2\alpha/p)})$$

and proceeding as in theorem 1, we prove this theorem.

Example 1 — Let $f_1(x, y) = [\pi(x + y)]^{-1/2}$ and let $\mu = \rho = \frac{5}{4}$ in the theorem, then we obtain

$$\phi_1(p, q) = \sqrt{(pq)}\,(\sqrt{p} + \sqrt{q})^{-1} \quad \text{(Ditkin and Prudnikov 1962, p. 127)}$$

$$\theta_1(p, q) = \frac{\pi}{4}\sqrt{(pq)}\,(\sqrt{p} + \sqrt{q})^{-2} \quad \text{(Ditkin and Prudnikov 1962, p. 128).}$$

$$\therefore \int_0^\infty \int_0^\infty (xy)^{-5/4}\,(x+y)^{-1/2}\, e^{\frac{1}{2}\left(\frac{\alpha^2}{x}+\frac{\beta^2}{y}\right)} D_{-5/2}\,(\alpha\sqrt{(2/x)})\, D_{-5/2}\,(\beta\sqrt{(2/y)})\, dx\,dy$$

$$= \tfrac{8}{9}\sqrt{(2\pi)}\,(\alpha\beta)^{-1/2}\,(\sqrt{\alpha} + \sqrt{\beta})^{-2}, \quad \text{Re}\,(\alpha, \beta) > 0.$$

Example 2 — Let $f_1(x, y) = (\pi y)^{-1/2} - \{\pi(x+y)\}^{-1/2}$, and let $\mu = \frac{3}{4}$, $\rho = \frac{7}{4}$ in the theorem, then we obtain

$$\phi_1(p, q) = q(\sqrt{p} + \sqrt{q})^{-1} \qquad \text{(Ditkin and Prudnikov 1962, p. 127)}$$

$$\theta_1(p, q) = \frac{\pi}{4} \sqrt{(pq)}\,(\sqrt{p} + \sqrt{q})^{-2} \quad \text{(Ditkin and Prudnikov 1962, p. 128).}$$

$$\therefore \int_0^\infty \int_0^\infty \left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{(x+y)}}\right) x^{-7/4}\, y^{-3/4}\, e^{\frac{1}{2}\left(\frac{\alpha^2}{x} + \frac{\beta^2}{y}\right)} D_{-7/2}(\alpha\sqrt{(2/x)})$$
$$\times D_{-3/2}(\beta\sqrt{(2/y)})\, dx\, dy$$
$$= \frac{8}{15} \sqrt{(2\pi)}\, (\alpha\beta)^{-1/2} (\sqrt{\alpha} + \sqrt{\beta})^{-2}, \ \text{Re}\,(\alpha, \beta) > 0.$$

Example 3 — Let $f_1(x, y) = \frac{1}{\pi} x^{1/2} y^{-1/2} (x+y)^{-1}$, and let $\mu = \frac{3}{4}$, $\rho = \frac{5}{4}$ in the theorem, then we obtain

$$\phi_1(p, q) = \sqrt{p}\, q(\sqrt{p} + \sqrt{q})^{-1} \qquad \text{(Ditkin and Prudnikov 1962, p. 127)}$$

$$\theta_1(p, q) = \frac{\pi}{8} \sqrt{(pq)}\,(\sqrt{p} + \sqrt{q})^{-2} \quad \text{(Ditkin and Prudnikov 1962, p. 128).}$$

$$\therefore \int_0^\infty \int_0^\infty x^{-3/4}\, y^{-5/4}\, (x+y)^{-1}\, e^{\frac{1}{2}\left(\frac{\alpha^2}{x} + \frac{\beta^2}{y}\right)} D_{-5/2}(\alpha\sqrt{(2/x)})\, D_{-3/2}(\beta\sqrt{(2/y)})\, dx\, dy$$
$$= \tfrac{4}{3}\, \pi(\alpha\beta)^{-1/2} (\sqrt{\alpha} + \sqrt{\beta})^{-2}, \ \text{Re}\,(\alpha, \beta) > 0.$$

§4. *Theorem* 3 — Let $f_1(x, y) \doteqdot \phi_1(p, q)$ ...(4.1)

$$x^{(\lambda/2-3/2)}\, y^{(\mu/2-3/2)}\, \phi_1(\sqrt{8x}, \sqrt{8y}) \doteqdot \theta_1(p, q) \qquad \text{...(4.2)}$$

then

$$\theta_1(\alpha, \beta) = \Gamma(\mu)\, \Gamma(\lambda)\, \alpha^{(1-\lambda/2)}\, \beta^{(1-\mu/2)}\, 2^{5\left(\frac{\mu}{2} + \frac{\lambda}{2} - 1\right)} \int_0^\infty \int_0^\infty f_1(x, y)$$
$$\times e^{\left(\frac{x^2}{\alpha} + \frac{y^2}{\beta}\right)} D_{-\lambda}(2x/\sqrt{\alpha})\, D_{-\mu}(2y/\sqrt{\beta})\, dx\, dy, \ \text{Re}\,(\alpha, \beta, \mu, \lambda) > 0 \qquad \text{...(4.3)}$$

provided $f_1(x, y)$ is absolutely integrable in $0 \leqslant x < \infty$, $0 \leqslant y < \infty$.

PROOF: This theorem can be proved by using the known result

$$x^{\lambda-1}\, e^{-1/8(x^2/\alpha)} \doteqdot p\Gamma(\lambda)\, 2^{\lambda}\, \alpha^{\lambda/2}\, e^{\alpha p^2}\, D_{-\lambda}(2p\sqrt{\alpha}).$$

Example — Let $f_1(x, y) = \dfrac{x^a + y^a}{\Gamma(a+1)}$ and let $\mu = \lambda = a + 3$ in the theorem, then we obtain

$$\phi_1(pq) = (pq)^{-a}(p^a + q^a), \text{ Re }(a) > -1 \text{ (Ditkin and Prudnikov 1962, p. 138)}$$

$$\theta_1(p, q) = 2^{-3a/2}\,\Gamma(a/2 + 1)\,(p^{a/2} + q^{a/2}) \text{ (Ditkin and Prudnikov 1962, p. 138).}$$

$$\therefore \int_0^\infty \int_0^\infty (x^a + y^a)\, e^{\left(\frac{x^2}{\alpha} + \frac{y^2}{\beta}\right)} D_{-a-3}(2x/\sqrt{\alpha})\, D_{-a-3}(2y/\sqrt{\beta})\, dx\, dy$$

$$= \sqrt{(\alpha\beta)}\,(\alpha^{a/2} + \beta^{a/2})\,(\sqrt{2})^{-(13a+20)}\,\{\Gamma(a+3)\}^{-2}\Gamma(a/2+1)\,\Gamma(a+1)$$

Re $(a) > -1$, Re $(\alpha, \beta) > 0$.

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