# ON SOME TRIPLE INTEGRALS INVOLVING MEIJER'S $G$-FUNCTION

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(Communicated by R. S. Mishra, F.N.A.)

(*Received* 3 *February* 1973)

Dahiya [1971, eqns. (1), (2)] has evaluated two double integrals in two dimensions $x$ and $y$ and involved a Meijer's $G$-function (Erdélyi 1953, p. 206) in the integrand. In this paper the authors have generalized Dahiya's results by evaluating four triple integrals in three dimensions $x, y$ and $z$ and involved a Meijer's $G$-function in the integrand. Many new results have been derived.

§1. We have established the following four integral relations :

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{-1/2}(x^2+y^2)^{-1/2\,(v+1)}\, y^v \cos\left(2u\tan^{-1}\frac{y}{x}\right)$$

$$\times f(x^2+y^2+z^2)\, F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right) G^{m,n}_{p,q}\left[\frac{a(x^2+y^2+z^2)^2}{y^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] dx\, dy\, dz$$

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}} \int_0^\infty \int_0^\infty (t^2+w^2)^{-1/2} f(t^2+w^2)\, F\left(\tan^{-1}\frac{w}{t}\right)$$

$$\times G^{m+2,n}_{p+2,q+2}\left[\frac{a(t^2+w^2)^2}{w^2}\,\middle|\,\begin{matrix}(a_p),\ \frac{1}{2}v+u+1,\ \frac{1}{2}v-u+1\\ \frac{1}{2}v+\frac{1}{2},\ \frac{1}{2}v+1,\ (b_q)\end{matrix}\right] dt\, dw \qquad \ldots(1.1)$$

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{-1/2}(x^2+y^2)^{-1/2\,(v+1)}\, x^v \cos\left(2u\tan^{-1}\frac{y}{x}\right)$$

$$\times f(x^2+y^2+z^2)\, F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right) G^{m,n}_{p,q}\left[\frac{a(x^2+y^2+z^2)^2}{x^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] dx\, dy\, dz$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty \int_0^\infty (t^2+w^2)^{-1/2} f(t^2+w^2)\, F\left(\tan^{-1}\frac{w}{t}\right)$$

$$\times G^{m+2,n}_{p+2,q+2}\left[\frac{a(t^2+w^2)^2}{w^2}\,\middle|\,\begin{matrix}(a_p),\ \frac{1}{2}v+u+1,\ \frac{1}{2}v-u+1\\ \frac{1}{2}v+\frac{1}{2},\ \frac{1}{2}v+1,\ (b_q)\end{matrix}\right] dt\, dw \qquad \ldots(1.2)$$

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2+z^2)^{-1/2}(x^2+y^2)^{-1/2(v+1)}y^v\cos\left(2u\tan^{-1}\frac{y}{x}\right)$$

$$\times f(x^2+y^2+z^2)\,F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\times G^{m,n}_{p,q}\left[\frac{a(x^2+y^2+z^2)^2(x^2+y^2)}{z^2y^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]dx\,dy\,dz$$

$$=\frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}}\int_0^\infty\int_0^\infty (t^2+w^2)^{-1/2}f(t^2+w^2)\,F\left(\tan^{-1}\frac{w}{t}\right)$$

$$\times G^{m+2,n}_{p+2,q+2}\left[\frac{a(t^2+w^2)^2}{t^2}\,\middle|\,\begin{matrix}(a_p),\,\frac{1}{2}v+u+1,\,\frac{1}{2}v-u+1\\\frac{1}{2}v+\frac{1}{2},\,\frac{1}{2}v+1\,(b_q)\end{matrix}\right]dt\,dw \quad \ldots(1.3)$$

and

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2+z^2)^{-1/2}(x^2+y^2)^{-1/2(v+1)}x^v\cos\left(2u\tan^{-1}\frac{y}{x}\right)$$

$$\times f(x^2+y^2+z^2)\,F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\times G^{m,n}_{p,q}\left[\frac{a(x^2+y^2+z^2)^2}{x^2}\,\frac{(x^2+y^2)}{z^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]dx\,dy\,dz$$

$$=\frac{\sqrt{\pi}}{2}\int_0^\infty\int_0^\infty (t^2+w^2)^{-1/2}f(t^2+w^2)\,F\left(\tan^{-1}\frac{w}{t}\right)$$

$$\times G^{m+2,n}_{p+2,q+2}\left[\frac{a(t^2+w^2)^2}{t^2}\,\middle|\,\begin{matrix}(a_p),\,\frac{1}{2}v+u+1,\,\frac{1}{2}v-u+1\\\frac{1}{2}v+\frac{1}{2},\,\frac{1}{2}v+1,\,(b_q)\end{matrix}\right]dt\,dw \quad \ldots(1.4)$$

provided $\mathrm{Re}(v)>0$, $|\arg a|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $p+q<2(m+n)$ and $f$, $F$ are two functions so that the integrals exist.

PROOF: To prove the result (1.1), we start with the integral

$$\int_0^{\frac{1}{2}\pi}\cos 2u\phi(\sin\phi)^v\,G^{m,n}_{p,q}\left[\frac{ar^2}{\sin^2\theta\sin^2\phi}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]d\phi$$

$$=\frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}}G^{m+2,n}_{p+2,q+2}\left[\frac{ar^2}{\sin^2\theta}\,\middle|\,\begin{matrix}(a_p),\,\frac{1}{2}v+u+1,\,\frac{1}{2}v-u+1\\\frac{1}{2}v+\frac{1}{2},\,\frac{1}{2}v+1,\,(b_q)\end{matrix}\right] \quad \ldots(1.5)$$

provided $\mathrm{Re}(v)>0$, $p+q<2(m+n)$ and $|\arg a|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$. In order to prove (1.5), we substitute the contour integral for the $G$-function and change the order of integration, which is permissible under the given conditions; the left-hand side of (1.5) equals

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)\,a^s\,r^{2s}}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)(\sin^{2s}\theta)}\,ds$$

$$\times\int_0^{\frac{1}{2}\pi}\cos 2u\,\phi(\sin\phi)^{v-2s}\,d\phi. \qquad \ldots(1.6)$$

Now evaluating the inner integral with the help of the formula (Sneddon 1956, p. 41) viz.

$$\int_0^{\frac{1}{2}\pi}\cos 2u\,\phi(\sin\phi)^{v}\,d\phi=\frac{\Gamma(v+1)\,\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2^{v+1}\,\Gamma(\frac{1}{2}v+u+1)\,\Gamma(\frac{1}{2}v-u+1)}, \qquad \ldots(1.7)$$

$$\mathrm{Re}\,(v)>0,$$

and then using the definition of $G$-function, we obtain the result (1.5).

On multiplying both sides of (1.5) by $f(r^2)\,F(\theta)\,dr\,d\theta$ and integrating $\theta$ between 0 to $\frac{1}{2}\pi$ and $r$ between 0 to $\infty$, we get

$$\int_0^{\infty}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{1}{r^2\sin\theta}\,f(r^2)\,F(\theta)\cos 2u\,\phi(\sin\phi)^v\,G_{p,q}^{m,n}\left[\frac{ar^2}{\sin^2\theta\,\sin^2\phi}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]$$

$$\times\, r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=\frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}}\int_0^{\infty}\int_0^{\frac{1}{2}\pi}\frac{1}{r}\,f(r^2)\,F(\theta)$$

$$\times\, G_{p+2,q+2}^{m+2,n}\left[\frac{ar^2}{\sin^2\theta}\,\middle|\,\begin{matrix}(a_p)\ \frac{1}{2}v+u+1,\ \frac{1}{2}v-u+1\\ \frac{1}{2}v+\frac{1}{2},\ \frac{1}{2}v+1,\ (b_q)\end{matrix}\right] r\,dr\,d\theta. \qquad \ldots(1.8)$$

Now using the spherical polar coordinates $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$ in the left-hand side of (1.8) where $r^2\sin\theta\,dr\,d\theta\,d\phi$ is the elementary volume $dx\,dy\,dz$; and polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ in the right-hand side where $r\,dr\,d\theta$ is the elementary area $dxdy$, we obtain the result (1.1).

Again to prove (1.2), we start with the integral

$$\int_0^{\frac{1}{2}\pi}\cos 2u\,\phi(\cos\phi)^v\,G_{p,q}^{m,n}\left[\frac{ar^2}{\sin^2\theta\,\cos^2\phi}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]d\phi$$

$$=\frac{\sqrt{\pi}}{2}\,G_{p+2,q+2}^{m+2,n}\left[\frac{ar^2}{\sin^2\theta}\,\middle|\,\begin{matrix}(a_p),\ \frac{1}{2}v+u+1,\ \frac{1}{2}v-u+1\\ \frac{1}{2}v+\frac{1}{2},\ \frac{1}{2}v+1,\ (b_q)\end{matrix}\right] \qquad \ldots(1.9)$$

provided $\mathrm{Re}(\nu) > 0$, $p + q < 2(m + n)$ and $|\arg a| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$. To prove (1.9), we substitute the contour integral for the $G$-function, changing the order of integration and applying the integral

$$\int_0^{\frac{1}{2}\pi} \cos 2u\,\phi (\cos\phi)^{\nu} = \frac{\pi\Gamma(\nu+1)}{2^{\nu+1}\,\Gamma(\frac{1}{2}\nu+u+1)\,\Gamma(\frac{1}{2}\nu-u+1)} \qquad \ldots(1.10)$$

$\mathrm{Re}(\nu) > 0$ and $u$ being an integer.

The result (1.10) is obtained by putting $\frac{1}{2}\pi - \phi$ in place of $\phi$ in the integral (1.7).

Now, multiplying both sides of (1.9) by $f(r^2)\,F(\theta)\,dr\,d\theta$ and integrating $\theta$ between 0 to $\frac{1}{2}\pi$ and $r$ between 0 to $\infty$, we get

$$\int_0^{\infty}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{1}{r^2\sin\theta} f(r^2)\,F(\theta)\cos 2u\,\phi(\cos\phi)^{\nu}$$

$$\times\, G^{m,n}_{p,q}\left[\frac{ar^2}{\sin^2\theta\cos^2\phi}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{\sqrt{\pi}}{2}\int_0^{\infty}\int_0^{\pi/2}\frac{1}{r} f(r^2)\,F(\theta)\,G^{m+2,n}_{p+2,q+2}\left[\frac{ar^2}{\sin^2\theta}\,\middle|\,\begin{matrix}(a_p),\frac{1}{2}\nu+u+1,\frac{1}{2}\nu-u+1\\ \frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+1,(b_q)\end{matrix}\right] r\,d\theta\,dr$$

$$\ldots(1.11)$$

Now, interpreting (1.11) as in (1.8), we arrive at the result (1.2). The integral equations (1.3) and (1.4) can be deduced in similar manners by taking $\cos^2\theta\,\sin^2\phi$ and $\cos^2\theta\,\cos^2\phi$ in the argument of $G$-function from the relations (1.5) and (1.9).

§2. *Particular Cases* — (i) On taking $F\left(\tan^{-1}\frac{w}{t}\right) = \cos\left(2\mu\tan^{-1}\frac{w}{t}\right)$

$$\times\,\sin\nu\left(\tan^{-1}\frac{w}{t}\right) = \left(\frac{w^2}{t^2+w^2}\right)^{(1/2)\nu}\cos\left(2\mu\tan^{-1}\left(\frac{w}{t}\right)\right)$$

and $f(t^2+w^2) = (t^2+w^2)^{1/2} f(t^2+w^2)$ in (1.1), we obtain

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} (x^2+y^2+z^2)^{\nu/2}\,(x^2+y^2)^{(1/2)(\nu-v-1)}\,y^{v}\cos\left(2u\tan^{-1}\frac{y}{x}\right)$$

$$\times\,\cos\left(2\mu\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right) f(x^2+y^2+z^2)$$

$$\times\, G^{m,n}_{p,q}\left[\frac{a(x^2+y^2+z^2)^2}{y^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right] dx\,dy\,dz =$$

*(equation continued on p. 1177)*

$$= \frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{2\sqrt{\pi}} \int_0^\infty \int_0^\infty \left(\frac{w^2}{t^2+w^2}\right)^{\nu/2} \cos\left(2\mu \tan^{-1}\frac{w}{t}\right) f(t^2+w^2)$$

$$\times\, G^{m+2,n}_{p+2,q+2}\left[\frac{a(t^2+w^2)^2}{w^2}\,\middle|\, \begin{matrix}(a_p),\, \frac{1}{2}v+u+1,\, \frac{1}{2}v-u+1\\ \frac{1}{2}v+\frac{1}{2},\, \frac{1}{2}v+1,\, (b_q)\end{matrix}\right] dx\, dy \qquad \ldots(2.1)$$

$$= \frac{\Gamma_*(\frac{1}{2}+u)\,\Gamma_*(\frac{1}{2}\pm\mu)}{8\pi} \int_0^\infty G^{m+4,n}_{p+4,q+4}$$

$$\times \left[a\eta \,\middle|\, \begin{matrix}(a_p),\, \frac{1}{2}\nu \pm u+1,\, \frac{1}{2}\nu \pm \mu+1\\ \frac{1}{2}\nu+\frac{1}{2},\, \frac{1}{2}\nu+1,\, \frac{1}{2}v+\frac{1}{2},\, \frac{1}{2}v+1,\, (b_q)\end{matrix}\right] f(\eta)\, d\eta, \text{ by Dahiya (1971)} \qquad \ldots(2.2)$$

provided $\mathrm{Re}\,(v) > 0$, $\mathrm{Re}\,(\nu) > 0$, $|\arg a| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\,\pi$ and $f(\eta)$ is such that the integral exists, and $\Gamma_*(a \pm b)$ stands for $\Gamma(a+b) \,.\, \Gamma(a-b)$.

Now, on taking $f(\eta) = \eta^{-\rho} e^{-\beta\eta}$, and so $f(x^2+y^2+z^2) = (x^2+y^2+z^2)^{-\rho} \times e^{-\beta\cdot(x^2+y^2+z^2)}$ and utilizing a known integral (Erdélyi 1954, p. 419), we obtain

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{-\rho-\nu/2}\,(x^2+y^2)^{1/2\,(\nu-v-1)}\, y^v\, e^{-\beta(x^2+y^2+z^2)}$$

$$\times \cos\left(2u \tan^{-1}\frac{y}{x}\right) \cos\left(2\mu \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\times\, G^{m,n}_{p,q}\left[\frac{a(x^2+y^2+z^2)^2}{y^2}\,\middle|\, \begin{matrix}(a_p)\\ (b_q)\end{matrix}\right] dx\, dy\, dz$$

$$= \frac{\Gamma_*(\frac{1}{2}\pm u)\,\Gamma_*(\frac{1}{2}\pm\mu)}{8\pi}\, \beta^{\rho-1}$$

$$\times\, G^{m+4,n+1}_{p+5,q+4}\left[\frac{a}{\beta}\,\middle|\, \begin{matrix}\rho,\, (a_p),\, \frac{1}{2}v \pm u+1,\, \frac{1}{2}\nu \pm \mu+1\\ \frac{1}{2}\nu+\frac{1}{2},\, \frac{1}{2}\nu+1,\, \frac{1}{2}v+\frac{1}{2},\, \frac{1}{2}v+1,\, (b_q)\end{matrix}\right] \qquad \ldots(2.3)$$

provided $p+q < 2(m+n)$, $|\arg a| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\,\pi$, $|\arg\beta| < \frac{1}{2}\pi$, $\mathrm{Re}\,(-\rho+\delta+1) > 0$, $\delta = \min \mathrm{Re}\,[b_h,\ (h=1,2,\ldots,m),\ \frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}\nu+1,\ \frac{1}{2}v+\frac{1}{2},\ \frac{1}{2}v+1]$, $\mathrm{Re}\,(v) > 0$ and $\mathrm{Re}\,(\nu) > 0$.

(ii) On taking the same substitution as in (i) above, in the integral equation (1.2), we obtain the integral

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{-\rho-\nu/2}\,(x^2+y^2)^{1/2(\nu-v-1)}\, x^v\, e^{-\beta(x^2+y^2+z^2)}$$

$$\times \cos\left(2u \tan^{-1}\frac{y}{x}\right) \cos\left(2\mu \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$

(*equation continued on p.* 1178)

$$\times\, G_{p,q}^{m,n}\left[\frac{a(x^2+y^2+z^2)^2}{x^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]dx\,dy\,dz$$

$$=\frac{\Gamma_*(\tfrac{1}{2}\pm\mu)\,\beta^{\rho-1}}{8}$$

$$\times\, G_{p+5,q+4}^{m+4,n+1}\left[\frac{a}{\beta}\,\middle|\,\begin{matrix}\rho,(a_p),\tfrac{1}{2}\nu\pm u+1,\tfrac{1}{2}\nu\pm\mu+1\\ \tfrac{1}{2}\nu+\tfrac{1}{2},\tfrac{1}{2}\nu+1,\tfrac{1}{2}\nu+1,\tfrac{1}{2}\nu+1,(b_q)\end{matrix}\right] \quad ...(2.4)$$

with the conditions given in (2.3).

(iii) On taking $f(t^2+w^2)=(t^2+w^2)^{1/2}\,f(t^2+w^2)$ and

$$F\left(\tan^{-1}\frac{w}{t}\right)=\cos\left(2\mu\tan^{-1}\frac{w}{t}\right)\cos\nu\left(\tan^{-1}\frac{w}{t}\right)$$
$$=\left(\frac{t^2}{t^2+w^2}\right)^{\nu/2}\cos\left(2\mu\tan^{-1}\frac{w}{t}\right)$$

in (1.3) and utilizing the result given by Dahiya [1971, eqn. (2)] we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2+z^2)^{-\rho-\nu/2}\,(x^2+y^2)^{-1/2\,(\nu+1)}\,y^\nu\,z^\nu\,e^{-\beta(x^2+y^2+z^2)}$$
$$\cos\left(2u\tan^{-1}\frac{y}{x}\right)\cos\left(2\mu\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$
$$\times\, G_{p,q}^{m,n}\left[\frac{a(x^2+y^2+z^2)^2\,(x^2+y^2)}{y^2z^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]dx\,dy\,dz$$
$$=\frac{\Gamma_*(\tfrac{1}{2}\pm u)}{8}\,.\,\beta^{\rho-1}\,.\,G_{p+5,q+4}^{m+4,n+1}\left[\frac{a}{\beta}\,\middle|\,\begin{matrix}\rho,(a_p),\tfrac{1}{2}\nu\pm u+1,\tfrac{1}{2}\nu\pm\mu+1\\ \tfrac{1}{2}\nu+\tfrac{1}{2},\tfrac{1}{2}\nu+1,\tfrac{1}{2}\nu+\tfrac{1}{2},\tfrac{1}{2}\nu+1,(b_q)\end{matrix}\right], \quad ...(2.5)$$

with the conditions given in (2.3).

(iv) On applying the same substitution, as in (iii) above, in (1.4) we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2+z^2)^{-\rho-\nu/2}\,(x^2+y^2)^{-1/2\,(\nu+1)}\,x^\nu\,z^\nu\,e^{-\beta(x^2+y^2+z^2)}$$
$$\cos\left(2u\tan^{-1}\frac{y}{x}\right)\cos\left(2\mu\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$
$$\times\, G_{p,q}^{m,n}\left[\frac{a(x^2+y^2+z^2)^2\,(x^2+y^2)}{x^2z^2}\,\middle|\,\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]dx\,dy\,dz$$
$$=\tfrac{1}{8}\,\beta^{\rho-1}\,.\,G_{p+5,q+4}^{m+4,n+1}\left[\frac{a}{\beta}\,\middle|\,\begin{matrix}\rho,(a_p),\tfrac{1}{2}\nu\pm u+1,\tfrac{1}{2}\nu\pm\mu+1\\ \tfrac{1}{2}\nu+\tfrac{1}{2},\tfrac{1}{2}\nu+1,\tfrac{1}{2}\nu+\tfrac{1}{2},\tfrac{1}{2}\nu+1,(b_q)\end{matrix}\right] \quad ...(2.6)$$

with the conditions given in (2.3).

(v) On taking $f(\eta) = \eta^{\rho-1} G_{k,l}^{f,g}\left[b\eta \,\middle|\, \begin{matrix} c_1, c_2, \ldots, c_k \\ d_1, d_2, \ldots, d_l \end{matrix}\right]$

in integral relation (2.2) and using the result given by Prasad (1969 p. 70) we obtain the integral

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2 + y^2 + z^2)^{\rho - \nu/2 - 1} (x^2 + y^2)^{1/2(\nu - v - 1)} y^v \cos\left(2u \tan^{-1} \frac{y}{x}\right)$$

$$\times \cos\left(2\mu \tan \frac{\sqrt{x^2 + y^2}}{z}\right) G_{k,l}^{f,g}\left[b(x^2 + y^2 + z^2) \,\middle|\, \begin{matrix} (c_k) \\ (d_l) \end{matrix}\right]$$

$$\times G_{p,q}^{m,n}\left[\frac{a(x^2 + y^2 + z^2)^2}{y^2} \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right] dx\, dy\, dz$$

$$= \frac{\Gamma_*(\frac{1}{2} \pm u)\, \Gamma_*(\frac{1}{2} \pm \mu)}{8} \int_0^\infty x^{\rho-1}\, G_{p+4,q+4}^{m+4,n}$$

$$\times \left[a\eta \,\middle|\, \begin{matrix} (a_p), \frac{1}{2}v \pm u + 1, \frac{1}{2}\nu \pm \mu + 1 \\ \frac{1}{2}v + \frac{1}{2}, \frac{1}{2}v + 1, \frac{1}{2}\nu + \frac{1}{2}, \frac{1}{2}\nu + 1, (b_q) \end{matrix}\right] \cdot G_{k,l}^{f,g}\left[b\eta \,\middle|\, \begin{matrix} (c_k) \\ (d_l) \end{matrix}\right] d\eta$$

$$= \frac{\Gamma_*(\frac{1}{2} \pm u)\, \Gamma_*(\frac{1}{2} \pm \mu)}{8} b^{-\rho}\, G_{p+l+4,q+k+4}^{m+g+4,n+f}\left[\frac{a}{b} \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_n \\ \frac{1}{2}v + \frac{1}{2}, \frac{1}{2}v + 1, \frac{1}{2}\nu + \frac{1}{2}, \end{matrix}\right.$$

$$\left.\begin{matrix} 1 - d_1 - \rho, 1 - d_2 - \rho, \ldots, 1 - d_l - \rho, a_{n+1}, a_{n+2}, \ldots, a_p, \frac{1}{2}v \pm u + 1, \frac{1}{2}\nu \pm \mu + 1 \\ \frac{1}{2}\nu + 1, b_1, b_2, \ldots, b_m, 1 - c_1 - \rho, 1 - c_2 - \rho, \ldots, 1 - c_k - \rho, b_{m+1}, b_{m+2}, \ldots, b_q \end{matrix}\right] \quad \ldots(2.7)$$

provided that $\mathrm{Re}(v) > 0$, $\mathrm{Re}(\nu) > 0$, $\beta + \beta' < \mathrm{Re}(-\rho) < \delta + \delta'$, $|\arg a| < \frac{1}{2}\lambda\pi$, $\lambda > 0$, $A > 0$, $|\arg b| < \frac{1}{2}\lambda'\pi$, $\lambda' > 0$, and $A' > 0$, where $A = q - p$, $\delta = \min \mathrm{Re}\,[b_h, (h = 1, \ldots, m), \frac{1}{2}\nu + \frac{1}{2}, \frac{1}{2}\nu + 1, \frac{1}{2}v + \frac{1}{2}, \frac{1}{2}v + 1]$, $\lambda = 2m + 2n - p - q$ $\beta = \max \mathrm{Re}\,[a_i - 1]$, $(i = 1, 2, \ldots, n)$, $\delta' = \max \mathrm{Re}\,(d_h)\ (h = 1, 2, \ldots, f)$, $\beta' = \max \mathrm{Re}\,(C_i - 1)$, $(i = 1, 2, \ldots, q)$, $\lambda' = 2f + 2g - k - l$ and $A' = l - k$.

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