PROLONGATION OF f-STRUCTURE TO TANGENT BUNDLE OF ORDER 2

by SARAMMA MATHAI, *Department of Mathematics, I. P. College for Women, Delhi*

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In this paper the author has studied 2nd lift of f-structure to tangent bundle of order 2 and obtained the integrability conditions in $T_2(M)$. Further f-invariant vector fields and incompressible vector fields in $T_2(M)$ are studied.

Let M be an n-dimensional differentiable manifold of class C^∞. Let $T_2(M)$ be the tangent bundle of order 2 over M, i.e. $T_2(M)$ is the set of all 2 jets of M and $T_2(M)$ is topologized. Then $T_2(M)$ is also a differentiable manifold of class C^∞ and its dimension is $3n$. The tangent bundle $T_1(M)$ of M is the set $T_1(M)$ of all 1-jets of M and $T_1(M)$ is topologized (Dombrowski 1962, Yano and Ishihara 1966, 1968 and Yano and Kobayashi 1966). $\pi_1 : T_1(M) \to M$ is the bundle projection of $T_1(M)$ to M and $\pi_2 : T_2(M) \to M$ is the bundle projection of $T_2(M)$ to M, and π_{12} is the mapping $\pi_{12} : T_2(M) \to T_1(M)$ (Yano and Ishihara 1968).

Thus

$$\pi_2 = \pi_1 \circ \pi_{12}.$$

Let U be a coordinate neighbourhood of M with (x^i) as coordinates. The indices $i, j, k, \ldots$ take values $\{1, 2, \ldots, n\}$. Thus every point Q belonging to the set $\pi_2^{-1}(U)$ of $T_2(M)$ has coordinates (x^i, y^i, z^i) which are called the coordinates induced in $\pi_2^{-1}(u)$ or simply induced coordinates. On putting $\xi^h = x^h$, $\xi^{\bar{h}} = y^h$, $\xi^{\bar{\bar{h}}} = z^h$ the induced coordinates are denoted by (ξ^A). The indices $A, B, C, \ldots$ take values

$$\{1, 2, \ldots n, \bar{1}, \bar{2}, \ldots \bar{n}, \bar{\bar{1}}, \bar{\bar{2}}, \ldots \bar{\bar{n}}\}.$$

Let $U(x^i)$ and $U'(x'^i)$ be two coordinate neighbourhoods of M such that $U \cap U' \neq 0$. Then $\pi_2^{-1}(U) \cap \pi_2^{-1}(U')$ is also not empty in $T_2(M)$.

Thus to the coordinate transformation

$$x^{a'} = x'^a(x^1, \ldots, x^n)$$

of $U \cap U'$ there corresponds the coordinate transformation

$$x'^h = x'^h(x^1, \ldots, x^n)$$

$$y'^h = \frac{\partial x'^h}{\partial x^h} y^h$$

$$z'^h = \frac{\partial x'^h}{\partial x^h} z^h + \frac{\partial^2 z'^h}{\partial x^j \partial x^i} y^j y^i$$

of $\pi_2^{-1}(U \cap U')$

f-structure in $T_2(M)$: A tensor field of type (1, 1) in M satisfying $f^3 + f = 0$ and of constant rank is called an f-structure in M (Yano 1961, 1963). Let $\tilde{f}^{II}$ denote the 2nd lift of f in M. Then

$$(\tilde{f}^{II})^3 + \tilde{f}^{II} = 0.$$

Let $\tilde{f}_B^A$ denote the components of $\tilde{f}^{II}$.

Accordingly (Yano and Ishihara 1968)

$$\left(\tilde{f}_B^A\right) = \begin{pmatrix} f_i^h & 0 & 0 \\ y^s \partial_s f_i^h & f_i^h & 0 \\ z^s \partial_s f_i^h + y^s y^t \partial_s \partial_t f_i^h & 2y^s \partial_s f_i^h & f_i^h \end{pmatrix}.$$

Let us define the following

$$\tilde{l}_B^A = -\tilde{f}_C^A \tilde{f}_B^C$$

and

$$\tilde{m}_B^A = -\tilde{l}_B^A + \delta_B^A$$

where δ_B^A takes the value unity when $A = B$ and vanishes otherwise.

Clearly the components of $\tilde{l}^{II}$ and $\tilde{m}^{II}$ are given by

$$\left(\tilde{l}_B^A\right) = \begin{pmatrix} l_j^i & 0 & 0 \\ y^s \partial_s l_j^i & l_j^i & 0 \\ z^s \partial_s l_j^i + y^t y^s \partial_t \partial_s l_j^i & 2y^s \partial_s l_j^i & l_j^i \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{m}_B^A \end{pmatrix} = \begin{pmatrix} m_j^i & 0 & 0 \\ y^s \partial_s m_j^i & m_j^i & 0 \\ z^s \partial_s m_j^i + y^t y^s \partial_t \partial_s m_j^i & 2y^s \partial_s m_j^i & m_j^i \end{pmatrix}$$

respectively. Which shows that $\tilde{l}_B^A$ and $\tilde{m}_B^A$ are respectively the 2nd lift of l_j^i and m_j^i, where l_j^i and m_j^i are the components of complementary projection tensors in M defined by $l = -f^2$ and $m = 1 + f^2$.

Further it can easily be verified that

$$\tilde{l}_A^E \; \tilde{l}_E^B = \tilde{l}_A^B$$

$$m_A^E \; \tilde{m}_E^B = \tilde{m}_A^B$$

Consequently we get the following Proposition.

Proposition 1 — The 2nd lift of l and m are complementary projection operators in $T_2(M)$. Thus there exists in $T_2(M)$ complementary distributions L_1 and M_1 corresponding to $\tilde{l}^{II}$ and $\tilde{m}^{II}$ respectively.

Further we get the following Proposition.

Proposition 2 — For an f-structure in $T_2(M)$ and the complementary projection operators $\tilde{l}^{II}$ and $\tilde{m}^{II}$ in $T_2(M)$ we have

$$\tilde{f}_A^E \; \tilde{l}_E^B = \tilde{l}_A^E \; \tilde{f}_E^B$$

$$\tilde{f}_E^A \; \tilde{f}_B^E \; \tilde{l}_D^B = -\tilde{l}_D^A$$

$$\tilde{f}_A^E \; \tilde{m}_E^B = \tilde{m}_A^E \; \tilde{f}_E^B = 0$$

and

$$\tilde{f}_E^A \; \tilde{f}_B^E \; \tilde{m}_D^B = 0$$

that is f^{II} acts on L_1 as an almost complex structure and on M_1 as a null operator.

Let ∇^{II} denote the lift in $T_2(M)$ of the connection ∇ in M. Then in view of proposition (7.1) (Yano and Ishihara 1968) for an arbitrary vector field ξ in M, we have

$$\nabla^{II} \xi^0 = (\nabla \xi)^0, \; \nabla^{II} \xi^I = (\nabla \xi)^I, \; \nabla^{II} \xi^{II} = (\nabla \xi)^{II}$$

Hence we get the following Proposition.

Proposition 3 — The zeroth lift, 1st lift and 2nd lift of a vector field ξ in M to the tangent bundle of order 2 $T_2(M)$ of M is incompressible if ξ is incompressible in M.

Let $L_{\tilde{X}}\ \tilde{f}^{II}$ denote the Lie derivative of the tensor $\tilde{f}^{II}$ with respect to the vector field $\tilde{X}$ in $T_2(M)$. Then

$$L_{\tilde{X}}\ \tilde{f}^{II}(\tilde{Y}) = [\tilde{X}, \tilde{f}^{II}\tilde{Y}] - \tilde{f}^{II}[\tilde{X}, \tilde{Y}]$$

where $[\tilde{X}, \tilde{Y}]$ is the Lie-bracket of $\tilde{X}$ and $\tilde{Y}$, $\tilde{X}$ and $\tilde{Y}$ being arbitrary vector fields in $T_2(M)$.

A vector field X in M is f-invariant with respect to the f-structure f if (Young-Bai Baik 1967)

$$L_X f = 0.$$

In particular let $\tilde{X}$ be the zeroth-lift of the vector field X in M and $\tilde{f}^{II}$ be the f-structure.

Then (Yano and Ishihara 1968)

$$L_{X^0}\tilde{f}^{II} = (L_X f)^0.$$

Thus we have the following.

Proposition 4 — The zeroth lift of a vector field X in M is f-invariant in $T_2(M)$ if and only if X is f-invariant in M.

Further in view of (Yano and Ishihara 1968)

$$L_{X^I}\tilde{f}^{II} = (L_X f)^I$$

and

$$L_{X^{II}}\tilde{f}^{II} = (L_X f)^{II}$$

we get the following Proposition.

Proposition 5 — If X is an f-invariant vector field in M with respect to the f-structure f, then 1st lift and 2nd lift of X are also f-invariant in $T_2(M)$ with respect to the f-structure $\tilde{f}^{II}$.

Integrability of f-structure in $T_2(M)$: Let $\tilde{N}$ denote the Nijenhuis tensor of the f-structure in $T_2(M)$. Then

$$\tilde{N}(\tilde{X}, \tilde{Y}) = [\tilde{f}^{II}\tilde{X}, \tilde{f}^{II}\tilde{Y}] - \tilde{f}^{II}[\tilde{f}^{II}\tilde{X}, \tilde{Y}] - \tilde{f}^{II}[\tilde{X}, \tilde{f}^{II}\tilde{Y}] + (\tilde{f}^{II})^2[\tilde{X}, \tilde{Y}]$$

The f-structure is said to be integrable if the Nijenhuis tensor vanishes.

Now we have the following result (Yano and Ishihara 1968)

$$\tilde{N} = N^{II}$$

where N is the Nijenhuis tensor of the f-structure f in M and N^{II} denotes the 2nd lift of N.

Thus we have the following Proposition.

Proposition 6 — If the f-structure f in M is integrable, then the f-structure $\tilde{f}^{II}$ in $T_2(M)$ is also integrable.

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