NOTE ON THE GENERATING FUNCTION FOR GENERALIZED HYPERGEOMETRIC POLYNOMIALS

by B. K. KARANDE, *Science College, Karad (Maharashtra)*

and

N. K. THAKARE, *Department of Mathematics, Shivaji University, Kolhapur (Maharashtra)*

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In this note the following theorem has been proved by a different method.

Theorem — The generalized hypergeometric polynomials

$$\overset{a}{\underset{n}{f}}(x) = \binom{a + (b+1)n}{n} {}_{p+1}F_{q+1}\left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ 1 + \alpha + bn, \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \; x\right]$$

are generated by

$$\sum_{n=0}^{\infty} \overset{a}{\underset{n}{f}}(x)\; t^n = \frac{(1+v)^{1+a}}{1 - bv}\, {}_pF_q\left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \; -xv\right]$$

where $v = t(v+1)^{b+1}$, p, q are non-negative integers, the α's and a, b take general values, real or complex, and $\beta_j \neq 0, -1, -2, \ldots, j = 1, 2, \ldots, q$.

As a consequence of this theorem we obtain the following result

$$\left\{ L_n^{a+nb}(x) \right\}$$

which is of Sheffer A-type zero for any complex number b, where $L_n^a(x)$ is a Laguerre polynomial.

§1. Brown (1969) proved the following lemma.

Lemma — Given a sequence φ_n $(n \geqslant 0)$, define the new one

$$\psi_n = \sum_{k=0}^{n} \binom{\alpha + \beta n}{n-k} \varphi_k, \; (n \geqslant 0). \qquad \ldots(1)$$

Then the following two results hold:

$$\sum_{n=0}^{\infty} \psi_n \left[\frac{v}{1+v}\right]^n = \frac{(1+v)^{1+\alpha}}{1 + (1-\beta)v} \sum_{n=0}^{\infty} \varphi_n v^n \qquad \ldots(2)$$

and

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha + \beta n} \psi_n \left[\frac{v}{(1+v)^{\beta}}\right]^n = (1+v)^{\alpha} \sum_{n=0}^{\infty} \frac{\alpha}{\alpha + \beta n} \varphi_n v^n. \quad \ldots(3)$$

Put $\alpha = a$, $v = t(v+1)^{b+1}$, where $\beta = b + 1$ in (1), (2) and (3) to get

$$\psi_n = \sum_{k=0}^{n} \binom{a + (b+1)n}{n-k} \varphi_k; \; (n \geqslant 0) \quad \ldots(4)$$

$$\sum_{n=0}^{\infty} \psi_n t^n = \frac{(1+v)^{1+a}}{1 - bv} \sum_{n=0}^{\infty} \varphi_n v^n \quad \ldots(5)$$

$$\sum_{n=0}^{\infty} \frac{a}{a + (b+1)n} \psi_n t^n = (1+v)^a \sum_{n=0}^{\infty} \frac{a}{a + (b+1)n} \varphi_n v^n. \quad \ldots(6)$$

respectively. Now we are in a position to prove the following theorem.

Theorem I — The generalized hypergeometric polynomials

$$f_n^a(x) = \binom{a + (b+1)n}{n} {}_{p+1}F_{q+1}\left[\begin{matrix} -n, \alpha_1, \alpha_2, \ldots, \alpha_p; \\ 1 + \alpha + bn, \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \; x\right] \quad \ldots(7)$$

are generated by

$$\sum_{n=0}^{\infty} f_n^a(x)\, t^n = \frac{(1+v)^{1+a}}{1 - bv} \, {}_pF_q\left[\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix} \; -xv\right] \quad \ldots(8)$$

where $v = t(v+1)^{b+1}$, p, q are non-negative integers, the α's and a, b take general values real or complex, and $\beta_j \neq 0, -1, -2, \ldots, j = 1, 2, 3, \ldots, q$.

PROOF : If we put

$$\varphi_n = \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_q)_n} \cdot \frac{(-x)^n}{n!}$$

in (4) then ψ_n becomes the generalized hypergeometric polynomial $f_n^a(x)$ of (7) and (5) becomes (8). This completes the proof.

The results of Theorem I are due to Srivastava (1969) but in our method of attack we are not required to use Laplace transform.

§2. Incidently we state that since the generalized Laguerre polynomials are defined by

$$L_n^a(x) = \binom{a+n}{n} {}_1F_1\left[\begin{matrix} -n; \\ 1+a; \end{matrix}\; x\right]$$

we have

$$L_n^{a+bn}(x) = \binom{a+(b+1)n}{n} {}_1F_1\left[\begin{matrix} -n; \\ 1+a+nb; \end{matrix}\; x\right]$$

and hence because of Theorem I we conclude that these polynomials $L_n^{a+nb}(x)$ are generated by

$$\sum_{n=0}^{\infty} L_n^{a+nb}(x)\, t^n = \frac{(1+v)^{a+1}}{1-bv} \exp(-xv) \qquad ...(9)$$

Note that b is any complex number. This result is originally due to Carlitz (1968).

It is well known that a polynomial set $\{p_n(x)\}$ is of Sheffer A-type zero if they are generated by

$$\sum_{n=0}^{\infty} p_n(x)\, t^n = \alpha(t) \exp[x\beta(t)] \qquad ...(10)$$

where it is understood that $\alpha(t)$ and $t^{-1}\beta(t)$ have at least formal power series expansions with non-zero initial coefficients. As $v = t(v+1)^{b+1}$ we conclude that the generating function (9) is of the type (10). Hence we get the following generalization of Brown's (1968) theorem.

Theorem II — $\left\{L_n^{a+nb}(x)\right\}$ is of Sheffer A-type zero for any complex number b.

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