ON THE STABILITY OF HETEROGENEOUS SHEAR FLOWS

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(Communicated by F. C. Auluck, F.N.A.)

(*Received* 25 *May* 1973)

The infinitesimal stability of inviscid, parallel, heterogeneous shear flows to two dimensional disturbances is considered when the Richardson number is everywhere non-negative. A criterion is established which implies that the principle of exchange of stabilities is not satisfied. Bounds for the phase velocity c_r of an arbitrary non-singular neutral mode are also obtained for velocity profiles which do not have any point of inflexion in the flow domain.

1. INTRODUCTION

The stability equation for infinitesimal disturbances in an inviscid, incompressible, heterogeneous, parallel shear flow, when the Boussinesq approximation is made, is

$$(U-c)^2 w'' - U''(U-c)\, w - k^2(U-c)^2 w + g\beta w = 0, \qquad \ldots(1)$$

where $w(z)$ is a single Fourier component of the vertical perturbation velocity, z is the vertical coordinate and primes denote differentiation with respect to z; $U(z)$ is the basic velocity profile and $c(=c_r+ic_i)$ is the complex phase speed of the wave mode of wavenumber k; g is the acceleration due to gravity and $\beta = -\rho'/\rho \geqslant 0$ everywhere in the flow domain. Together with suitable boundary conditions, it defines an eigenvalue problem for c given k or vice versa. The boundary conditions to be considered here are that w vanishes on $z = z_1$ and z_2 (rigid walls) which may recede to $\pm\infty$ in the limiting cases. Some of the important results which are relevant to the present work are the following :

(i) The complex phase speed of any unstable mode (i.e., with $c_i > 0$) must lie in a semi-circle in the c plane which has the range of U for diameter (Howard 1961).

(ii) A sufficient condition for stability is that $g\beta - \left(\frac{U'^2}{4}\right)$ should be everywhere non-negative (Miles 1961, Howard 1961).

(iii) The existence of a singular neutral mode, i.e. a mode with $c_i = 0$ and c_r within the range of U, implies the existence of contiguous unstable modes (Miles 1963).

(iv) The principle of 'exchange of stabilities' holds for a stationary singular neutral mode (i.e. $c_r = 0 = c_i$) if $U'(z)$ and $\beta'(z)$ are positive definite functions of U that possess analytic continuation into the complex U plane (Miles 1963).

The above theorems have formed the basis of many previous analytic investigations. The determination of the stability boundary of any given configuration in the wavenumber Richardson number (a typical one) plane and a further characterization of it in the sense whether the principle of 'exhange of stabilities' is satisfied or not is one of the most important physical aspects of the problem. This aspect is only partially covered by Theorems (iii) and (iv). Theorem (iii) shows that if a singular neutral mode exists then at least one unstable mode also exists although nothing can be said in general as to whether the associated neutral curve is a stability boundary or otherwise. Theorem (iv) which appears to remove this difficulty in the case of stationary singular neutral modes is rather weak in its application because of its restrictions on U' and β' as a function of U. Only a few configurations satisfy these conditions. In the present paper we establish a criterion which implies that stationary singular neutral modes cannot exist and hence the principle of 'exchange of stabilities' is not satisfied whether the associated neutral curve is a stability boundary or otherwise. Further, the problem of determining the bounds for the eigenvalues of c which correspond to a non-singular neutral mode, i.e. $c_i = 0$ with c_r not lying in the range of U is also not covered by Theorems (i) and (ii). We obtain here an upper and a lower bound for such eigenvalues of c in the respective cases when U'' is positive or negative everywhere in the flow domain.

2. Perturbation Equations

The initial stationary state whose stability we wish to examine is that of an incompressible inviscid fluid of continuously varying density in which there is horizontal streaming. To be specific, we shall suppose that the streaming takes place in the x-direction with velocity U. The assumption that the fluid is inviscid allows us to consider ρ and U as arbitrary functions of the height z.

Let the actual density at any point (x, y, z) as the result of a disturbance be $\rho + \delta\rho$. Let the corresponding change in the pressure be δp, and finally, let the components of the velocity in the perturbed state be $U + u$, v and w. Then the equations governing the perturbations are

$$\rho \frac{\partial u}{\partial t} + \rho U \frac{\partial u}{\partial x} + \rho w \frac{dU}{dz} = -\frac{\partial \delta p}{\partial z} \qquad \text{...(2)}$$

$$\rho \frac{\partial v}{\partial t} + \rho U \frac{\partial v}{\partial x} = -\frac{\partial \delta p}{\partial y} \qquad \text{...(3)}$$

$$\rho \frac{\partial w}{\partial t} + \rho U \frac{\partial w}{\partial x} = -\frac{\partial \delta p}{\partial z} - g\delta\rho \qquad \text{...(4)}$$

$$\frac{\partial \delta\rho}{\partial t} + U\frac{\partial \delta\rho}{\partial x} = - w\frac{d\rho}{dz} \qquad ...(5)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad ...(6)$$

Analysing the disturbance into normal modes, we seek solutions whose dependance on x, y and t is given by

$$\exp\,[i(k_x x + k_y y + nt)] \qquad ...(7)$$

where

$$k = \sqrt{k_x^2 + k_y^2} \qquad ...(8)$$

is the wavenumber of the disturbance, k_x and k_y being real, and n is a constant which can be complex.

For solutions having this dependence on x, y and t, eqns. (2)–(6) become

$$i\rho(n + Uk_x)\,u + \rho(DU)\,w = - ik_x\delta p, \qquad ...(9)$$

$$i\rho(n + Uk_x)\,v = - ik_y\delta p, \qquad ...(10)$$

$$i\rho(n + Uk_x)\,w = - D\delta p - g\delta\rho, \qquad ...(11)$$

$$i(n + Uk_x)\delta\rho = - wD\rho, \qquad ...(12)$$

and

$$i(k_x u + k_y v) = - Dw, \qquad ...(13)$$

where D stands for d/dz.

Multiplying eqns. (9) and (10) by $-ik_x$ and $-ik_y$ respectively, adding and making use of eqn. (13), we obtain

$$i\rho(n + Uk_x)\,Dw - i\rho k_x(DU)\,w = - k^2\delta p \qquad ...(14)$$

whereas by combining eqns. (11) and (12), we have

$$i\rho(n + Uk_x)\,w = - D\delta p - ig(D\rho)\,\frac{w}{(n + Uk_x)}. \qquad ...(15)$$

Now eliminating δp between eqns. (14) and (15) and rearranging the resulting equation, we get

$$(n + Uk_x)\,(D^2 - k^2)\,w - k_x(D^2U)\,w - gk^2\,\frac{D\rho}{\rho}\,\frac{w}{n + Uk_x}$$

$$+ \frac{D\rho}{\rho}\,[(n + Uk_x)\,Dw - k_x(DU)\,w] = 0. \qquad ...(16)$$

The last two terms on the left-hand side of eqn. (16) represent respectively the effects of heterogeneity of the fluid on potential energy and inertia. In most of the cases of interest, the neglect of this latter effect, in comparison with the former is justified (see for example, Drazin 1958, Chandrasekhar 1961 etc.). In this framework, eqn. (16) can be simplified in the form

$$(U - c)^2 D^2 w - (U - c) D^2 U w - k^2 (U - c)^2 w + g\beta w = 0 \qquad ...(17)$$

where

$$c = - n/k, \qquad ...(18)$$

$$\beta = - \frac{D\rho}{\rho} \geqslant 0 \text{ everywhere in the flow domain,} \qquad ...(19)$$

and $k_x = k$ (since these are the most destabilizing disturbances for this problem, see Drazin 1958).

If we suppose that the fluid is confined between $z = 0$ and $z = d$ (say), then we must require that the solutions of eqn. (17) must satisfy the boundary conditions

$$w = 0 \text{ at } z = 0 \text{ and } z = d. \qquad ...(20)$$

Equations (17) – (20) present an eigenvalue problem for c, the other quantities being prescribed, and the system is unstable, neutral or stable for a given disturbance according as $c_i \neq 0$, $c_i = 0$ with $(U - c_r)$ vanishing at least once in the flow domain and $c_i = 0$ with $(U - c_r)$ vanishing nowhere in the flow domain respectively, c_r and c_i being the real and imaginary parts of c. Further, if $c_i = 0$ implies $c_r = 0$ for all $k > 0$ then the principle of exchange of stabilities is satisfied.

3. Auxilliary Theorem

The real part of the definite integral

$$I_1 = \int_0^d f(z)\, w^* w'' dz \qquad ...(21)$$

where w^* is the complex conjugate of w and $f(z)$ is a real-valued continuous function of z having continuous derivatives at least up to second order is given by

$$\text{Re}\, I_1 = \tfrac{1}{2} \int_0^d |w|^2 f'' dz - \int_0^d |w'|^2 f\, dz, \qquad ...(22)$$

Re I_1 being the real part of I_1 and primes denote differentiation with respect to z.

Proof: Integrating I_1 by parts and using the boundary conditions (20), it follows that

$$I_1 = -\int_0^d f'w^*w'dz - \int_0^d f \mid w' \mid^2 dz. \qquad ...(23)$$

Integrating the first integral on the right-hand side of eqn. (23) by parts and using the boundary conditions (20), we have

$$\int_0^d f'w^*w'dz = -\int_0^d f'' \mid w \mid^2 dz - \int_0^d f'w^{*\prime}w\, dz. \qquad ...(24)$$

Now combining eqns. (23) and (24) we get the required result (see also, Banerjee *et al.* 1973).

Theorem 3.1—For the problem given by eqns. (17) - (20), the integral relation

$$-\int_0^d \left[\left\{(U-c_r)^2 + c_i^2\right\}^2 + 4\,c_i^2\,(U-c_r)^2\right]\left[\mid w' \mid^2 + k^2 \mid w^2\right] dz$$

$$-\; c_i^2 \int_0^d g\beta \mid w \mid^2 dz + \int_0^d \left[(U-c_r)^2\left\{6U'^2 + U''(U-c_r) + g\beta\right\}\right.$$

$$\left. +\; c_i^2\left\{2U'^2 + U''(U-c_r)\right\}\right] \mid w \mid^2 dz = 0 \qquad ...(25)$$

is true.

PROOF : Equation (17) can be written as

$$\left[\left\{(U-c_r)^2 - c_i^2\right\} - 2ic_i(U-c_r)\right]w'' - U''\left[(U-c_r) - ic_i\right]w$$

$$-\,k^2\left[\left\{(U-c_r)^2 - c_i^2\right\} - 2ic_i(U-c_r)\right]w + g\beta w = 0. \qquad ...(26)$$

Multiplying eqn. (26) by $\left[\left\{(U-c_r)^2 - c_i^2\right\} + 2ic_i(U-c_r)\right] w^*$ throughout and integrating the resulting equation over the vertical range of z, we get

$$\int_0^d \left[\left\{(U-c_r)^2 - c_i^2\right\}^2 + 4c_i^2\,(U-c_r)^2\right] w^*w''dz - \int_0^d U''\left\{(U-c_r) - ic_i\right\}$$

$$\times\left[\left\{(U-c_r)^2 - c_i^2\right\} + 2ic_i(U-c_r)\right] \mid w \mid^2 dz\, -$$

(*equation continued on p.* 1198)

$$- k^2 \int_0^d \left[\left\{ (U - c_r)^2 - c_i^2 \right\}^2 + 4c_i^2 (U - c_r)^2 \right] | w |^2 \, dz$$

$$+ \int_0^d g\beta \left[\left\{ (U - c_r)^2 - c_i^2 \right\} + 2ic_i(U - c_r) \right] | w |^2 \, dz = 0 \quad \text{...(27)}$$

or

$$I_1 - I_2 - I_3 + I_4 = 0 \quad \text{...(28)}$$

where I_1, I_2, I_3 and I_4 are respectively the first, second, third and the fourth integrals on the left-hand side of eqn. (27).

Now, using eqn. (22) we have

$$\text{Re}\, I_1 = \int_0^d \left[6U'^2(U - c_r)^2 + 2c_i^2 \, U'^2 + 2U''(U - c_r)^3 + 2c_i^2 \, U''(U - c_r) \right] | w |^2 \, dz$$

$$- \int_0^d \left[\left\{ (U - c_r)^2 - c_i^2 \right\}^2 + 4c_i^2 (U - c_r)^2 \right] | w' |^2 \, dz \quad \text{...(29)}$$

$$\text{Re}\, I_2 = \int_0^d \left[U''(U - c_r)^3 + c_i^2 \, U''(U - c_r) \right] | w |^2 \, dz \quad \text{...(30)}$$

$$\text{Re}\, I_3 = k^2 \int_0^d \left[\left\{ (U - c_r)^2 - c_i^2 \right\}^2 + 4c_i^2 (U - c_r)^2 \right] | w |^2 \, dz \quad \text{...(31)}$$

and

$$\text{Re}\, I_4 = \int_0^d g\beta \left\{ (U - c_r)^2 - c_i^2 \right\} | w |^2 \, dz. \quad \text{...(32)}$$

Equating the real parts of both sides of eqn. (27) and making use of eqns. (29) – (32), we have, after a little calculation, the required result, namely

$$- \int_0^d \left[\left\{ (U - c_r)^2 + c_i^2 \right\}^2 + 4c_i^2 (U - c_r)^2 \right] \left[| w' |^2 + k^2 | w |^2 \right] dz -$$

(equation continued on p. 1199)

$$- c_i^2 \int_0^d g\beta \, | w |^2 \, dz + \int_0^d \Big[(U - c_r)^2 \Big\{ 6U'^2 + U''(U - c_r) + g\beta \Big\}$$

$$+ c_i^2 \Big\{ 2U'^2 + U''(U - c_r) \Big\} \Big] \, | w |^2 \, dz = 0 \qquad ...(33)$$

Theorem 3.2 — A necessary condition for the existence of normal modes for the problem is that the expression

$$6U'^2 + U''(U - c_r) + g\beta \qquad ...(34)$$

is positive at least at one point z_s (say) in the flow domain.

PROOF : Let us assume that normal modes exist for the problem. This means that eqn. (33) is true since it holds for an arbitrary mode. However, it is clear that if condition (34) is violated everywhere in the flow domain then eqn. (33) cannot be satisfied. Hence the theorem.

Theorem 3.3 — Principle of 'exchange of stabilities' is not satisfied if

$$6U'^2 + UU'' + g\beta \leqslant 0 \qquad ...(35)$$

everywhere in the flow domain.

PROOF : For principle of 'exchange of stabilities' to be satisfied we must have $c_i = 0$ implying $c_r = 0$ for all $k > 0$. Then eqn. (33) which is true for an arbitrary mode gives that

$$- \int_0^d U^4 [\, | w' |^2 + k^2 \, | w |^2] \, dz + \int_0^d U^2 [6U'^2 + UU'' + g\beta] \, | w |^2 \, dz = 0. \qquad ...(36)$$

But eqn. (36) cannot be satisfied if condition (35) is true everywhere in $[0, d]$. Hence the theorem.

4. An Example

In the following analysis we shall construct an example of a velocity and density profile which satisfies condition (35) and hence for such a heterogeneous shear layer the marginal state must be overstable. Further, it is clear from the definition of singular neutral modes that we must have $U(z)$ vanishing at least once in the flow domain, for otherwise the modes represented by $c_i = 0 = c_r$ will imply non-singular neutral modes.

To solve the differential inequality (35) everywhere in the flow domain, we take

$$g\beta = \nu^2 U'^2 \qquad ...(37)$$

where

$$\nu^2 = \text{constant} < 1/4. \quad \ldots(38)$$

Condition (38) is to be necessarily satisfied because otherwise according to theorem (ii) of §1, the flow would be stable and hence the question of marginal modes will not arise at all. Thus, we shall find a solution for $U(z)$ which satisfies the differential inequality

$$UU'' + (6 + \nu^2) \leqslant 0 \quad \ldots(39)$$

everywhere in the flow domain.

To solve the above non-linear differential inequality we consider the non-linear differential equation

$$UU'' + \mu^2 U'^2 = 0 \quad \ldots(40)$$

where

$$\mu^2 = \text{an even integer} > (6 + \nu^2). \quad \ldots(41)$$

Clearly, the solution $U(z)$ of equation (40) will satisfy condition (39) everywhere in the flow domain as $\mu^2 > (6 + \nu^2)$. The reason why μ^2 is taken to be an even integer will be clear later on.

Multiplying both sides of eqn. (40) by U^{μ^2-1}, we get

$$U^{\mu^2-1}[UU'' + \mu^2 U'^2] = 0$$

which can be written as

$$[U^{\mu^2} U']' = 0. \quad \ldots(42)$$

Integrating eqn. (42) twice we have

$$U^{\mu^2+1} = c_1(1 + \mu^2)\, z + c_2 \quad \ldots(43)$$

where c_1 and c_2 are constants which are to be determined by the boundary conditions to be satisfied by $U(z)$ on $z = 0$ and $z = d$. As the fluid considered is inviscid $U(z)$ can take arbitrary values on $z = 0$ and $z = d$ and since $U(z)$ must necessarily vanish at least once in the flow domain, we take the boundary conditions on $U(z)$ as

$$U = -U_1^2 \text{ at } z = 0$$

and

$$U = U_2^2 \text{ at } z = d \quad \ldots(44)$$

where U_1 and U_2 are real constants having the dimensions of the square root of velocity.

Making use of the boundary condition (44), we have from eqn. (43)

$$U^{\mu^2+1} = \left[U_1^{2(\mu^2+1)} + U_2^{2(\mu^2+1)} \right] \frac{z}{d} - U_1^{2(\mu^2+1)}. \qquad \ldots(45)$$

The above velocity distribution $U(z)$ given by equation (45) represents a continuous function of z in $[0, d]$ (since μ^2 is an even integer) and vanishes at

$$z = z_r = d \; \frac{U_1^{2(\mu^2+1)}}{U_1^{2(\mu^2+1)} + U_2^{2(\mu^2+1)}} \qquad \ldots(46)$$

such that $0 < z_r < d$. Clearly, for this velocity profile, the differential inequality (39) will be satisfied everywhere in the flow domain. Hence with $U(z)$ given by eqn. (45) and $\rho(z)$ by eqn. (37) we cannot have $c_i = 0$ implying $c_r = 0$ for any $k > 0$ and thus the principle of 'exchange of stabilities' is not satisfied. However, the following point is to be carefully noted in this connection : the quantities U'', UU'' and U' are respectively unbounded at $z = z_r$ but still, the conclusion that principle of 'exchange of stabilities' is not satisfied holds good. This is because if eqns. (17) - (20) with $c_i = 0 = c_r$ and $U(z)$ and $\rho(z)$ given by eqns. (45) and (37) do not allow any regular solution for w in $[0, d]$ the above conclusion is trivially satisfied. If however eqns. (17) - (20) does allow a regular solution for w everywhere in $[0, d]$ so that each term in eqn. (17) remains bounded in $[0, d]$, we have by eqn. (36) that $w \equiv 0$ and hence the principle of 'exchange of stabilities' is not satisfied. We now show that, in this situation, overstable modes do exist at the marginal state. We prove the following theorems :

Theorem 4.1—The point $z = z_r$ is a regular singular point of the differential equation

$$(U - c)^2 w'' - U''(U - c) w - k^2(U - c)^2 w + g\beta w = 0, \qquad \ldots(47)$$

where $c = c_r \neq 0$ and $\rho(z)$, $U(z)$ and z_r respectively given by eqns. (37), (45) and (46).

PROOF : Using eqn. (37), we can write eqn. (47) as

$$(U - c)^2 w'' - U''(U - c) w - k^2(U - c)^2 w + \nu^2 U'^2 w = 0 \qquad \ldots(48)$$

or

$$P(z) w'' + Q(z) w' + R(z) w = 0 \qquad \ldots(49)$$

where

$$P(z) = (U - c)^2 \qquad \ldots(50)$$

$$Q(z) \equiv 0 \qquad \ldots(51)$$

and

$$R(z) = -[U''(U-c) + k^2(U-c)^2 - \nu^2 U'^2]. \qquad \ldots(52)$$

Now using eqns. (45), (46) and the fact that $c = c_r \neq 0$, we have

$$\lim_{z \to z_r} (z - z_r) \frac{Q(z)}{P(z)} = 0 \qquad \ldots(53)$$

and

$$\lim_{z \to z_r} (z - z_r)^2 \frac{R(z)}{P(z)} = 0. \qquad \ldots(54)$$

Hence, the point $z = z_r$ which clearly is a singular point of the differential equation (48) is a regular singular point.

Theorem 4.2 — In the neighbourhood of the regular singular point $z = z_r$, eqn. (48) allows two linearly independent regular solutions which are also regular at $z = z_r$.

PROOF: To establish this we make use of Frobenius' extended power series method. Let the solution be represented in the form

$$w(z) = (z - z_r)^s \sum_{n=0}^{\infty} a_n (z - z_r)^n$$

$$= \sum_{n=0}^{\infty} a_n (z - z_r)^{n+s} \qquad \ldots(55)$$

where s and a_n's are real constants which are to be determined and $a_0 \neq 0$ We then have

$$w''(z) = \sum_{n=0}^{\infty} a_n (n+s)(n+s-1)(z - z_r)^{n+s-2}. \qquad \ldots(56)$$

Now inserting these expressions for $w(z)$ and $w''(z)$, respectively, from eqns. (55) and (56) in eqn. (48), we get

$$\left[K_1 Z^{1/\mu^2+1} - c \right]^2 \sum_{n=0}^{\infty} a_n (n+s)(n+s-1) Z^{n+s-2} - K_2 Z^{-\frac{2\mu^2+1}{\mu^2+1}} \times$$

(*equation continued on p.* 1203)

$$\times\left[K_1Z^{1/\mu^2+1}-c\right]\sum_{n=0}^{\infty}a_nZ^{n+s}-k^2\left[K_1Z^{1/\mu^2+1}-c\right]^2$$

$$\times\sum_{n=0}^{\infty}a_nZ^{n+s}+\nu^2K_3Z^{-\frac{2\mu^2}{\mu^2+1}}\sum_{n=0}^{\infty}a_nZ^{n+s}=0 \qquad ...(57)$$

where

$$Z=z-z_r \qquad ...(58)$$

$$K_1=\left[\frac{U_1^{2(\mu^2+1)}+U_2^{2(\mu^2+1)}}{d}\right]^{1/\mu^2+1} \qquad ...(59)$$

$$K_2=-\frac{K_1\mu^2}{(\mu^2+1)^2} \qquad ...(60)$$

and

$$K_3=\frac{K_1^2}{(\mu^2+1)^2}. \qquad ...(61)$$

Equating to zero the coefficient of the lowest power of $(z-z_r)$, i.e., of Z^{s-2} in equation (57), we get the indicial equation

$$s(s-1)=0. \qquad ...(62)$$

The roots $s=0$ and $s=1$ of eqn. (62) differ by an integer and hence corresponding to the larger root, there corresponds a solution of eqn. (48), of the form

$$w_1(z)=\sum_{n=0}^{\infty}a_n(z-z_r)^{n+1} \qquad ...(63)$$

where the coefficients a_n's can be successively determined.

Corresponding to the smaller root $s=0$, we take a solution of the form

$$w_2(z)=Kw_1(z)\log Z+\sum_{n=0}^{\infty}b_nZ^n, \qquad ...(64)$$

where K and b_n's are constants which are to be determined. We have from the above

$$w_2'(z)=\frac{Kw_1(z)}{Z}+Kw_1'(z)\log Z+\sum_{n=0}^{\infty}nb_nZ^{n-1} \qquad ...(65)$$

and

$$w_2''(z) = -\frac{Kw_1(z)}{Z^2} + \frac{2Kw_1'(z)}{Z} + Kw_1''(z)\log Z + \sum_{n=0}^{\infty} n(n-1)\, b_n Z^{n-2}. \quad ...(66)$$

Now, since $w_2(z)$ is a solution of eqn. (48), we have by inserting the above values of $w_2(z)$ and $w_2''(z)$ in eqn. (48) and making use of the equation

$$(U-c)^2\, w_1'' - U''(U-c)\, w_1 - k^2(U-c)^2\, w_1 + \nu^2 U'^2 w_1 = 0$$

we get

$$-\frac{K}{Z^2}\left[K_1 Z^{1/\mu^2+1} - c\right]^2 \sum_{n=0}^{\infty} a_n Z^{n+1} + \frac{2K}{Z}\left[K_1 Z^{1/\mu^2+1} - c\right]^2$$

$$\times \sum_{n=0}^{\infty} a_n (n+1)\, Z^n + \left[K_1 Z^{1/\mu^2+1} - c\right]^2$$

$$\times \sum_{n=0}^{\infty} b_n n(n-1)\, Z^{n-2} - K_2 Z^{-\frac{2\mu^2+1}{\mu^2+1}} \left[K_1 Z^{1/\mu^2+1} - c\right]$$

$$\times \sum_{n=0}^{\infty} b_n Z^n - k^2 \left[K_1 Z^{1/\mu^2+1} - c\right]^2$$

$$\sum_{n=0}^{\infty} b_n Z^n + \nu^2 K_3 Z^{-\frac{2\mu^2}{\mu^2+1}} \sum_{n=0}^{\infty} b_n Z^n = 0. \quad ...(67)$$

Equating to zero the coefficient of $1/Z$ on the left-hand side of equation (67), by using the fact that μ^2 is an even integer greater than 6, we have

$$Kc^2 a_0 = 0. \quad ...(68)$$

This implies that

$$K = 0 \text{ (since } c \neq 0 \text{ and } a_0 \neq 0). \quad ...(69)$$

Hence the second solution $w_2(z)$ is given by

$$w_2(z) = \sum_{n=0}^{\infty} b_n Z^n, \quad ...(70)$$

where the coefficients b_n's can be successively determined. Clearly, the solutions $w_1(z)$ and $w_2(z)$ which are respectively given by eqns. (63) and (70) are linearly independent and regular everywhere in $[0,d]$ including the point $z = z_r$. This proves the theorem.

We have thus shown that for $U(z)$ given by equation (45) and $\rho(z)$ by eqn. (37) (which can be shown to yield a well defined density function $\rho(z)$), the principle of 'exchange of stabilities' is not satisfied and the marginal state is definitely overstable.

5. Bounds for the Phase Velocity c_r

We now derive bounds for the phase velocity c_r of an arbitrary non-singular neutral mode for velocity profiles which do not have any point of inflexion in the flow domain. We prove the following theorem :

Theorem 5.1 — For velocity profiles $U(z)$ which do not have any point of inflexion in the flow domain, the phase velocity c_r of an arbitrary non-singular neutral mode satisfies

$$c_r < b + \left[\frac{6U'^2 + g\beta}{|U''|}\right]_{\substack{\max\\ [0,\, d]}} \text{ provided } U'' > 0 \text{ everywhere in } [0,\, d] \quad ...(71)$$

or

$$c_r > a - \left[\frac{6U'^2 + g\beta}{|U''|}\right]_{\substack{\max\\ [0,\, d]}} \text{ provided } U'' < 0 \text{ everywhere in } [0,\, d] \quad ...(72)$$

where $a = U_{\min}$ and $b = U_{\max}$.

Proof : We have already established in Theorem 3.2 that a necessary condition for the existence of an arbitrary normal mode is that there must exist at least one point $z = z_s$ in the flow domain such that

$$[6U'^2 + U''(U - c_r) + g\beta]_{z=z_s} > 0. \quad ...(73)$$

Therefore, condition (73) also holds good for an arbitrary non-singular neutral mode, i.e., a mode for which $c_i = 0$ with $(U - c_r)$ not vanishing anywhere in the flow domain. It then clearly follows that

$$c_r < b + \left[\frac{6U'^2 + g\beta}{|U''|}\right]_{\substack{\max\\ [0.\, d]}} \text{ provided } U'' > 0 \text{ everywhere in } [0,\, d],$$

or

$$c_r > a - \left[\frac{6U'^2 + g\beta}{|U''|}\right]_{\substack{\max\\ [0,\, d]}} \text{ provided } U'' < 0 \text{ everywhere in } [0,\, d],$$

which gives the required result. In fact the above bounds hold good for any arbitrary mode whether unstable, non-singular neutral or singular neutral. But they contribute only for determining the bounds for the phase velocity c_r of arbitrary non-singular neutral modes as according to Miles (1961) result, the phase velocity c_r of an arbitrary unstable or singular neutral mode must necessarily satisfy $a < c_r < b$. We now evaluate these bounds, for a prescribed configuration.

An Example

Consider the velocity and density profiles given respectively by

$$U = U_0 e^{z/d} \qquad \ldots(74)$$

and

$$g\beta = c_1 U'^2 \qquad \ldots(75)$$

in $0 \leqslant z \leqslant d$, where U_0 is a constant (> 0) having the dimensions of U and c_1 is a dimensionless constant $> 1/4$ so that according to Miles' theorem the above constitutes a stable system. Thus, only non-singular neutral modes exist in the present case. Clearly, $U'' > 0$ everywhere in $[0, d]$ and hence theorem 5.1 gives

$$c_r < U_0 e\,[7 + c_1] \qquad \ldots(39)$$

Similarly we illustrate the result contained in inequality (72).

ACKNOWLEDGEMENT

The authors are grateful to Prof. P. L. Bhatnagar for his valuable remarks and for his kindness in reading the original manuscript.

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