CONDUCTIVE HEAT TRANSFER BETWEEN TWO CONCENTRIC SPHERES WITH INTERNAL DEGREES OF FREEDOM

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Heat transfer between two concentric spheres for a gas with internal degrees of freedom has been studied with the aid of Maxwell moment method, ultilizing a two-sided Maxwellian type distribution as weighting function. Analytical solutions for six moment approximations have been obtained for the heat transfer and translational and internal temperatures.

INTRODUCTION

In the case of polyatomic gas, the molecular energy consists of not only of the energy of translational motion but also contains some terms which are due to internal degrees of freedom (rotational, vibrational etc.). The first step in the investigation of kinetic equations for the gas with internal degrees of freedom was made by Wang-Chang and Uhlenbeck (1951). In this theory the effect of internal degree of freedom is taken into account by quantum mechanical method. Recently, a new equation was proposed by Morse (1964) which permits a kinetic description of polyatomic gas possessing internal degrees of freedom. The collision term in this model is related to Wang-Chang and Uhlenbeck (1951) results for polyatomic gas much in the same manner as Bhatnagar-Gross-Krooks (1954) model is related to the Boltzmann equation. For steady case the model equation is as follows (Morse 1964 and Hsu and Morse 1967) :

$$\xi_k \frac{\partial f_l}{\partial x_k} = \frac{M_t - f_l}{\tau_{el}} + \frac{M_i - f_l}{\tau_{in}} = J \qquad \ldots(1.1)$$

with

$$M_t = \eta_l(eq)\left(\frac{1}{2\pi RT_t}\right)^{3/2} \exp\left[-\frac{\vec{C}^2}{2RT_t}\right] \qquad \ldots(1.2)$$

$$M_i = \eta_l(eq)\left(\frac{1}{2\pi RT_i}\right)^{3/2} \exp\left[-\frac{\vec{C}^2}{2RT_i}\right] \qquad \ldots(1.3)$$

and

$$\eta_l(eq) = \frac{n \exp\left[-\frac{E_l}{RT_i}\right]}{\sum_l \exp\left[-\frac{E_l}{RT_i}\right]} \qquad ...(1.4)$$

where T_t and T_i are the translational and internal temperature, $\eta_l(eq)$ is the equilibrium density and E_l the energy for quantum state corresponding to level l. The first term in equation (1.1) contains the contribution due to elastic collision (τ_{el}) and the second due to inelastic collision (τ_{in})

Lees and Liu (1962) and Lees (1965) dealt with the problems of the heat transfer from a fine wire located along the axis of a cylindrical bell jar and between two concentric spheres. They utilized the Maxwell moment method by introducing "two sided Maxwellian" type distribution as the weighting function (Lees 1959). By using four moments in the Maxwell moment method they obtained the solutions to these problems valid over the whole range of gas densities from the Knudsen to the Fourier regimes.

In this paper we use the Maxwell moment method to study the problem of heat transfer between two concentric spheres with internal degrees of freedom. Six moments are considered and analytical solutions are obtained for heat transfer and translational and internal temperatures.

2. Formulation of the Problem

Consider two concentric spheres of radius r_1 and r_2 with $r_2 > r_1$. The inner sphere is kept at temperature Tw_1 while the outer sphere is kept at temperature Tw_2 and the annular region is filled with diatomic gas. Six parametric functions are required in order to satisfy the three relevant conservation equations plus three collision moment equations. The simplest choice of weight function, having a two-sided Maxwellian character and capable of giving a smooth transition between the highly rarefied gas regime and continuum limit, is as follows (Lees 1965) :

$$f = f_1 = \frac{\eta_1(eq)}{2\pi RT_{t_1}} \exp\left[-\frac{1}{2RT_{t_1}}\left(v_r^2 + v_\theta^2 + v_\phi^2\right)\right] \quad \text{for } 0 < \phi < \pi/2 - \alpha \qquad ...(2.1)$$

and

$$f = f_2 = \frac{\eta_2(eq)}{2\pi RT_{t_2}} \exp\left[-\frac{1}{2RT_{t_2}}\left(v_r^2 + v_\theta^2 + v_\phi^2\right)\right] \quad \text{for } \pi/2 - \alpha < \phi < \pi \qquad ...(2.2)$$

with

$$\eta_1(eq) = \frac{n_1 \exp\left\{-\frac{E_l}{RT_{i1}}\right\}}{\sum_l \exp\left\{-\frac{E_l}{RT_{i1}}\right\}} \qquad ...(2.3)$$

and

$$\eta_2(eq) = \frac{n_2 \exp\left\{-\frac{E_l}{RT_{i_2}}\right\}}{\sum_l \exp\left\{-\frac{E_l}{RT_{i_2}}\right\}} \qquad \text{...(2.4)}$$

where $\phi = \tan^{-1}\frac{v_\theta}{v_r}$ is the conical angle of the spherical geometry and $\alpha = \cos^{-1}\frac{r_1}{r}$. $n_1(r)$, $n_2(r)$, $T_{t1}(r)$, $T_{t2}(r)$, $T_{i1}(r)$ and $T_{i2}(r)$ are the six unknown functions of radial distance. Here n's have the dimension of the number density while the T_t's and T_i's have the dimension of translational and internal temperatures respectively. Knowing the distribution function f, we can evaluate all the six unknown quantities by averaging over all of velocity space:

$$\langle Q \rangle = \sum_l \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2-\alpha} Q f_1 \, dv_r \, dv_\theta \, dv_\phi + \sum_l \int_0^\infty \int_0^{2\pi} \int_{\pi/2-\alpha}^{\pi} Q f_2 \, dv_r \, dv_\theta \, dv_\phi$$

For example, the density is

$$n = \sum_l \eta_l = \sum_l \int f d\vec{v}$$

or

$$n = \tfrac{1}{2}\left[n_1(1 - \sin\alpha) + n_2(1 + \sin\alpha)\right]. \qquad \text{...(2.5)}$$

The translational and internal temperatures T_t and T_i, respectively, are

$$\frac{3}{2} RT_t = \sum_l \int f \frac{v^2}{2} d\vec{v} = \frac{3}{4}\left[n_1 T_{t1}(1 - \sin\alpha) + n_2 T_{t2}(1 + \sin\alpha)\right] \qquad \text{...(2.6)}$$

and

$$C_v^i T_i = \sum_l \int E_l f d\vec{v} = \frac{A}{2}\left[n_1 T_{i1}(1 - \sin\alpha) + n_2 T_{i2}(1 + \sin\alpha)\right] \qquad \text{...(2.7)}$$

where C_v^i is the internal specific heat at constant volume and A is any constant.

The total heat flux is given by

$$q = \sum_{l} \int \left(\frac{v^2}{2} + E_l\right) v_r f d\vec{v}$$

$$= \frac{\cos^2 \alpha}{(2\pi)^{1/2}} [n_1(RT_{t_1})^{1/2} (2RT_{t_1} + ART_{i_1}) - n_2(RT_{t_2})^{1/2} (2RT_{t_2} + ART_{i_2})] \quad \ldots(2.8)$$

3. Differential Equations and Boundary Conditions

For the spherical geometry, the Maxwell integral equation of transfer is as follows (Lees 1965):

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \int f v_r Q \, d\vec{v}\right] - \int \frac{f}{r} \left\{\left(v_\theta^2 + v_\phi^2\right) \frac{\partial Q}{\partial v_r}\right.$$

$$+ \left(\frac{\cos\theta}{\sin\theta} v_\phi^2 - v_\theta v_r\right) \frac{\partial Q}{\partial v_\theta} - \left(v_\phi v_r + \frac{\cos\theta}{\sin\theta} v_\phi v_\theta\right)$$

$$\left.\times \frac{\partial Q}{\partial v_\phi}\right\} d\vec{v} = \triangle Q \quad \ldots(3.1)$$

where Q is any function of the component of the particle velocity and $\triangle Q$ is the rate of change of Q produced by particle-particle collisions. Setting $Q = 1$, v_r, $(v^2 + E_l)$ successively, the mass, momentum and energy equations are obtained and we find $\triangle Q = 0$ because the conservative equations are invariant during collisions. Taking $Q = E_l$, $v^2 v_r$ and $E_l v_r$ successively, we obtain the equations of collision moment.

In order to bring out all pertinent parameters governing the problem, we choose n_0, Tw_1 and r_1 as the characteristic number density, temperature and length respectively. The additional parameters appearing in the problem are defined as

$$z = \frac{\tau_{el}}{\tau_{in}} \text{ and } \tau_{el} = \frac{\delta}{v_{th}} \quad \ldots(3.2)$$

where v_{th} is the thermal velocity, $\delta = \dfrac{\mu}{n_0 \left(\dfrac{2RTw_1}{\pi}\right)^{1/2}}$ is the maxwell mean free path and τ_{in} and τ_{el} are the relaxation time for inelastic and elastic collisions respectively.

Substituting the weighting function defined in equations (2.1) and (2.2) into six moment equations, we obtain six ordinary differential equations governing six unknown functions :

Conservative Equations

$$n_1(RT_{t_1})^{1/2} = n_2(RT_{t_2})^{1/2} \quad \ldots(3.3)$$

$$\sin^3\alpha \frac{d}{dr}[n_1(RT_{t_1}) - n_2(RT_{t_2})] + \frac{d}{dr}[n_1(RT_{t_1}) + n_2(RT_{t_2})] = 0 \quad ...(3.4)$$

$$n_1(RT_{t_1})^{1/2}[(RT_{t_1}) + A(RT_{i_1})] - n_2(RT_{t_2})^{1/2}[(RT_{t_2}) + A(RT_{i_2})] = \bar{\beta} \quad ...(3.5)$$

where $\bar{\beta}$ is the integration constant proportional to the heat flux.

Collision Moment Equations

$$\frac{1}{r^2}\frac{d}{dr}[n_1(RT_{t_1})^{1/2}(RT_{i_1}) - n_2(RT_{t_2})^{1/2}(RT_{i_2})] = \frac{1}{A}\sum_l \langle E_l J\rangle \quad ...(3.6)$$

$$\frac{d}{dr}[n_1(RT_{t_1}) + n_2(RT_{t_2})] + \sin^3\alpha \frac{d}{dr}[n_1(RT_{t_1}) - n_2(RT_{t_2})] = \sum_l \langle v^2 v_r J\rangle$$

...(3.7)

$$\frac{d}{dr}[n_1(RT_{t_1})(RT_{i_1}) + n_2(RT_{t_2})(RT_{i_2})] - \sin^3\alpha \frac{d}{dr}[n_1(RT_{t_1})(RT_{i_1}) - n_2(RT_{t_2})(RT_{i_2})] = \frac{2}{A}\sum_l \langle E_l v_r J\rangle \quad ...(3.8)$$

with

$$\frac{1}{A}\sum_l \langle E_l J\rangle = \frac{1}{A\tau_{in}}\left[n_1(1-\sin\alpha)\left\{\frac{1}{3+2A}\left(\frac{3}{2}(RT_{t_1}) + A(RT_{i_1})\right) - A(RT_{i_1})\right\} + n_2(1-\sin\alpha)\left\{\frac{1}{3+2A}\left(\frac{3}{2}(RT_{t_2}) + A(RT_{i_1})\right) - \frac{A(RT_{i_2})}{2}\right\}\right] \quad ...(3.9)$$

$$\sum_l \langle v^2 v_r J\rangle = -\left(\frac{1}{\tau_{el}} + \frac{1}{\tau_{in}}\right)[n_1(RT_{t_1})^{3/2} - n_2(RT_{t_2})^{3/2}] \times \frac{2}{\sqrt{\pi}}\cos^2\alpha \quad ...(3.10)$$

$$\sum_l \langle E_l v_r J\rangle = -A\left(\frac{1}{\tau_{in}} + \frac{1}{\tau_{el}}\right)\frac{\cos^2\alpha}{2\sqrt{\pi}}[n_1(RT_{t_1})^{1/2}(RT_{i_1}) - n_2(RT_{t_2})^{1/2}(RT_{i_2})] \quad ...(3.11)$$

where J is collisional integral defined in eqn. (1.1).

For diffuse remission the boundary conditions are, at $r = r_1$

$$T_{t_1} = T_{i_1} = 1 \quad ...(3.12a)$$

$$n_1 = 1 \quad \text{...(3.12b)}$$

and at $r = r_2$

$$T_{t_2} = T_{i_2} = \frac{Tw_2}{Tw_1}. \quad \text{...(3.12c)}$$

We shall work out the problem in the linearized form by introducing the following linearizations into eqns. (3.3) – (3.8), in which $Tw_2/Tw_1 = 1 - \epsilon$

$$n_1 = n_0(1 + \bar{n}_1),\ n_2 = n_0(1 + \bar{n}_2)$$

$$T_{t_1} = RTw_1(1 + \bar{T}_{t_1}),\ T_{t_2} = RTw_1(1 + \bar{T}_{t_2})$$

$$T_{i_1} = RTw_1(1 + \bar{T}_{i_1}),\ T_{i_2} = RTw_1(1 + \bar{T}_{i_2})$$

we obtain

$$\bar{T}_{t_1} - \bar{T}_{t_2} = 2(\bar{n}_2 - \bar{n}_1) \quad \text{...(3.13)}$$

$$\sin^3 \alpha \frac{d}{dr}(\bar{n}_2 - \bar{n}_1) + \frac{d}{dr}(\bar{n}_1 + \bar{n}_2 + \bar{T}_{t_1} + \bar{T}_{t_2}) = 0 \quad \text{...(3.14)}$$

$$(\bar{T}_{t_1} - \bar{T}_{t_2}) + (\bar{T}_{i_1} - \bar{T}_{i_2}) = \frac{\bar{\beta}}{n_0(RTw_1)^{3/2}} = \beta \quad \text{...(3.15)}$$

$$\frac{1}{r^2}\frac{d}{dr}(\bar{T}_{i_1} - \bar{T}_{i_2}) = \lambda z\left[\frac{3}{10}\left\{(\bar{T}_{t_1} + \bar{T}_{t_2}) - (\bar{T}_{i_1} + \bar{T}_{i_2})\right\}\right.$$
$$\left. - \sin\alpha\left\{\frac{3}{10}(\bar{T}_{t_1} - \bar{T}_{t_2}) - (\bar{T}_{i_1} - \bar{T}_{i_2})\right\}\right] \quad \text{...(3.16)}$$

$$\sin^3 \alpha \frac{d}{dr}(\bar{n}_2 - \bar{n}_1) + \frac{d}{dr}\left\{(\bar{n}_1 + \bar{n}_2) + 2(\bar{T}_{t_1} + \bar{T}_{2t})\right\}$$
$$= -2\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1 + z)(\bar{n}_2 - \bar{n}_1)\cos^2\alpha \quad \text{...(3.17)}$$

$$\frac{d}{dr}(\bar{n}_1 + \bar{n}_2 + \bar{T}_{t_1} + \bar{T}_{t_2} + \bar{T}_{i_1} + \bar{T}_{i_2}) - \sin^2\alpha\,(\bar{n}_2 - \bar{n}_1 + \bar{T}_{i_1} - \bar{T}_{i_2})$$
$$= -\lambda(1 + z)\sqrt{\left(\frac{2}{\pi}\right)}(\bar{T}_{i_1} - \bar{T}_{i_2})\cos^2\alpha \quad \text{...(3.18)}$$

where $\epsilon, \bar{n}_1, \bar{n}_2, \bar{T}_{t_1}, \bar{T}_{t_2}, \bar{T}_{i_1}, \bar{T}_{i_2} \ll 1$ and

$$\lambda = \frac{1}{\sqrt{(RTw_1)}\,\tau_{el}}$$ and for diatomic gas the constant $A = 1$.

In fact there is no difficulty in solving the set of above nonlinear equations. Using the boundary conditions (3.12), we can determine all the six unknown quantities in terms of β,

Simplifying eqns. (3.13) – (3.18), we obtain the following relations :

$$(\bar{T}_{t_1} - \bar{T}_{t_2}) = (\bar{T}_{i_1} - \bar{T}_{i_2}) = 2(\bar{n}_2 - \bar{n}_1) = \frac{\beta}{2} \qquad ...(3.19)$$

$$(\bar{T}_{t_1} + \bar{T}_{t_2}) = (\bar{T}_{i_1} + \bar{T}_{i_2}) = \lambda(1+z)\sqrt{\left(\frac{2}{\pi}\right)}\frac{\beta}{2}\frac{r_1}{\bar{r}} + C_1 \qquad ...(3.20)$$

and

$$\bar{n}_1 + \bar{n}_2 = -\lambda(1+z)\sqrt{\left(\frac{2}{\pi}\right)}\frac{\beta}{2}\frac{r_1}{\bar{r}} + C_2 \qquad ...(3.21)$$

where $\bar{r} = \frac{r}{r_1}$ and the constants C_1 and C_2 are determined by applying the boundary conditions

$$\bar{T}_{t_1} = \bar{T}_{i_1} = \bar{n}_1 = 0 \text{ at } \bar{r} = 1 \qquad ...(3.22)$$

$$\bar{T}_{t_2} = \bar{T}_{i_2} = -\epsilon \quad \text{at } \bar{r} = \frac{r_2}{r_1} = \bar{r}_2. \qquad ...(3.23)$$

$\bar{T}$'s and $\bar{n}$'s are then determined from eqns. (3.19) – (3.23)

$$\bar{T}_{t_1} = \bar{T}_{i_1} = \frac{\beta}{4}\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1+z)\,r_1\left(\frac{r}{\bar{r}} - 1\right) \qquad ...(3.24)$$

$$\bar{T}_{t_2} = \bar{T}_{i_2} = -\frac{\beta}{2}\left[1 - \frac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1+z)\,r_1\left(\frac{1}{\bar{r}} - 1\right)\right] \qquad ...(3.25)$$

$$\bar{n}_1 = -\frac{\beta}{4}\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1+z)\,r_1\left(\frac{1}{\bar{r}} - 1\right) \qquad ...(3.26)$$

$$\bar{n}_2 = \frac{\beta}{4}\left[1 - \sqrt{\left(\frac{2}{\pi}\right)}\lambda(1+z)\,r_1\left(\frac{1}{\bar{r}} - 1\right)\right. \qquad ...(3.27)$$

with

$$\beta = \frac{\epsilon}{\frac{1}{2} + \frac{1}{4}\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1+z)\,r_1\left(1 - \frac{r_2}{r_1}\right)}. \qquad ...(3.28)$$

Using the subscript kn to denote quantities evaluated at the free molecular limit (where the inverse of the Knudsen number tends to zero), we find then from equation (2.5), the heat flux ratio as

$$\frac{q}{q_{kn}} = \frac{2}{1 + \left\{\frac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1+z)\,r_1\left(1 - \frac{r_2}{r_1}\right)\right\}} \qquad ...(3.29)$$

Inserting the expressions for $\bar{T}$'s and $\bar{n}$'s from eqns. (3.24) - (3.27) into eqns. (2.6) and (2.7) for translational and internal temperatures, we obtain

$$\frac{Tw_1 - \bar{T}_t}{Tw_1 - Tw_2} = 1 + \frac{\Upsilon}{8}\left(1 + \sin\alpha\right) \qquad \ldots(3.30)$$

and

$$\frac{Tw_1 - \bar{T}_i}{Tw_1 - Tw_2} = \frac{R}{C_v^i}\left[1 + \frac{\Upsilon}{8}\left(1 + \sin\alpha\right)\right] \qquad \ldots(3.31)$$

with

$$\Upsilon = \frac{2}{1 + \frac{1}{2}\sqrt{\left(\frac{2}{\pi}\right)}\lambda(1 + z)\, r_1\left(1 - \frac{r_2}{r_1}\right)}. \qquad \ldots(3.32)$$

Conclusion

A kinetic description of heat transfer between two concentric spheres has been obtained by employing model obtained by Morse (1964) or Boltzmann equation with internal degrees of freedom by utilizing the Maxwell moment method. The analytical solutions obtained in eqns. (3.29) - (3.32) depend upon the ratio of elastic and inelastic collision $\left(\text{i.e., } \frac{\tau_{el}}{\tau_{in}}\right)$. Further, the analytic solutions show that the heat transfer consisted of spatially dependent contribution from the internal and translational energy modes. For $z \to 0$ (frozen internal degrees), we recover the results with BGK-model.

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