THE LEADING EDGE SIGNAL ON A FLAT PLATE IN UNIFORM SHEAR

by R. J. COLE, *Mathematics Department, University of Strathclyde, Glasgow G1 1XH, Scotland*

(Communicated by F. C. Auluck, F.N.A.)

(*Received* 16 *April* 1974)

A simple solution has been sought of the equation that governs the impulsively generated flow of a viscous incompressible fluid where the mainstream has uniform shear.

The method, which linearizes the boundary layer equation by making use of the Oseen approximation, expresses the solution as a series in powers of a small parameter, the ratio of external to boundary layer vorticity. The solution leads to a formula for the position of the leading edge signal which shows that the effect of a positive mainstream shear is to curve the signal downstream. The formula is also applied to an asymmetric flow profile.

1. INTRODUCTION

This paper concerns the impulsive motion, along its length, of a semi-infinite flat plate immersed in an unbounded stream of incompressible viscous fluid that contains shear.

It relates to two other well known theoretical problems. The first problem deals with the impulsive motion of a semi-infinite flat plate in a fluid at rest at infinity. Stewartson (1951, 1960) has studied this problem and describes it as a 'Rayleigh problem'. But for the viscous boundary condition at the plate the solution would be the undisturbed flow.

This problem features the existence of a leading edge signal which separates the flow region that is independent of t from the flow region that is independent of x. When we linearize the boundary layer equation (Stewartson 1951) the leading edge signal takes on the particularly simple form

$$x = u_0 t \qquad \ldots(1.1)$$

where x measures the distance downstream from the leading edge of the plate, u_0 represents the speed of the undisturbed stream relative to the plate and t is the time measured from the impulsive start. For $x < u_0 t$ the solution is independent of t and for $x > u_0 t$ the solution is independent of x.

The following question arises. If the undisturbed stream possessed 'shear' how would the leading edge signal be modified? The possibility of external shear leads us to consider the second related problem which is concerned with the 'steady' flow past a semi-infinite flat plate of a fluid that contains external vorticity. Ferri and Libby (1954) raised the question of the interaction between the boundary-layer and the external flow. They pointed out that the boundary between regions where shear stresses are and are not important is not well-defined. Much of the discussion has centred on whether or not the interaction produces a streamwise pressure gradient (Li 1955, 1956; Glauert 1957, 1962). Murray (1961) found one, in the case of unbounded shear, by matching inner and outer expansions. However, Toomre and Rott (1964) showed that the pressure level at the plate is strongly influenced by the boundedness of the shear. Mark (1962, 1966) examined an asymmetric situation in which the flows above and below the plate differ from each other.

Our object, in considering the impulsive interaction problem, lies in finding an expression for the leading edge signal. We expect the signal equation (1.1) to be modified by the presence of external vorticity. For simplicity we assume that the streamwise pressure gradient is zero in the interaction layer. We also linearize the boundary layer equation. From the solution to the linearized equation we can extract an expression for the leading edge signal. The solution is restricted to the case in which the ratio of the external vorticity to the boundary layer vorticity is small. Essentially it forms a perturbation of the uniform stream case. But a straightforward series expansion of the solution in powers of the parameter $\omega z/u_0$, where z is the co-ordinate normal to the plate and where the undisturbed stream u is given by

$$u = u_0 + \omega z \qquad \text{...(1.2)}$$

fails to modify the leading edge signal (1.1). The series fails because the signal equation is determined by the 'zeroth' approximation, which neglects the external shear entirely. We can overcome this difficulty by employing the technique of co-ordinate straining. In doing so we improve the range of validity of the new series and at the same time introduce into the zeroth approximation a form for the leading edge signal that depends upon x, t and z.

2. Formulation of the Problem

The boundary layer equations for two-dimensional incompressible flow are, assuming constant pressure,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad \text{...(2.1)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} \qquad \text{...(2.2)}$$

where u, w are the components of fluid velocity in the directions x and z respectively, and ν the coefficient of viscocity. The plate lies in the plane $z = 0$, $x \geqslant 0$. Suitable boundary conditions are

$$\left.\begin{aligned} u(0, z, t) &= u_0 + \omega z, \;\; t \geqslant 0, \;\; z > 0 \\ u(x, z, 0) &= u_0 + \omega z, \;\; x \geqslant 0, \;\; z > 0 \\ u(x, 0, t) &= 0, x > 0, \;\; t > 0 \\ \frac{\partial u}{\partial z}(x, z, t) &\to \omega, \;\; z \to \infty, x \geqslant 0, \;\; t \geqslant 0. \end{aligned}\right\} \qquad \ldots(2.3)$$

If the external flow were uniform, the Oseen linearization would amount to replacing the non-linear convective terms in (2.2) by the term $u_0 \partial u/\partial x$. We take account of the mainstream shear in our linearization by substituting the term $(u_0 + \omega z)\, \partial u/\partial x$. The governing equation now becomes

$$\frac{\partial u}{\partial t} + \left(u_0 + \omega z\right) \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial z^2} = 0 \qquad \ldots(2.4)$$

an equation in the single dependent variable u. (We can calculate the normal velocity component w from the equation of continuity (2.1)).

Define non-dimensional parameters ϵ, X, Z, T, U by the following transformations :

$$\frac{\omega z}{u_0} = Z\epsilon \qquad \ldots(2.5)$$

$$\frac{\omega^2 \nu x}{u_0^3} = X\epsilon^2 \qquad \ldots(2.6)$$

$$\frac{\omega^2 \nu t}{u_0^2} = T\epsilon^2 \qquad \ldots(2.7)$$

$$u = u_0 + \omega z - u_0 U. \qquad \ldots(2.8)$$

In this set ϵ is an artificial small parameter. If we define X to be of order unity in the region of interest then (2.6) can be re-expressed as

$$\frac{(\nu x/u_0)^{1/2}}{u_0/\omega} = O(\epsilon) \ll 1. \qquad \ldots(2.9)$$

Thus ϵ measures the ratio of the boundary layer thickness over which the streamwise velocity changes by u_0 to the vortical length over which the mainstream velocity changes by u_0. To keep ϵ small we must therefore consider stations not too far downstream. Within this region the interaction between the boundary layer and external vorticities is weak.

We now substitute (2.5) – (2.9) into (2.4) and (2.3) and obtain

$$\frac{\partial U}{\partial T} + \left(1 + \epsilon Z\right)\frac{\partial U}{\partial X} - \frac{\partial^2 U}{\partial Z^2} = 0 \qquad \text{...(2.10)}$$

$$\left.\begin{array}{lll} U(0, Z, T) = 0, & T \geqslant 0, \; Z > 0 \\ U(X, Z, 0) = 0, & X \geqslant 0, \; Z > 0 \\ U(X, 0, T) = 1, & X > 0, \; T > 0 \\ \dfrac{\partial U}{\partial Z} \to 0, \; Z \to \infty, & X \geqslant 0, \; T \geqslant 0. \end{array}\right\} \qquad \text{...(2.11)}$$

We wish to solve equation (2.10) subject to boundary and initial conditions (2.11).

3. Method of Solution

We look for a solution, which is a perturbation about the basic Rayleigh result, in the form

$$U(X, Z, T) = U_0(X, Z, T) + \epsilon U_1(X, Z, T) + \epsilon^2 U_2(X, Z, T) + \ldots . \qquad \text{...(3.1)}$$

By substituting (3.1) into (2.10) and equating coefficients in successive powers of ϵ we find

$$LU_0 = 0, \qquad \text{...(3.2)}$$

$$LU_n = -Z\frac{\partial U_{n-1}}{\partial X}, \; n = 1, 2, \ldots, \qquad \text{...(3.3)}$$

where

$$L \equiv \frac{\partial}{\partial T} + \frac{\partial}{\partial X} - \frac{\partial^2}{\partial Z^2} \qquad \text{...(3.4)}$$

is the Rayleigh operator. Corresponding boundary conditions for the U_n can be obtained on substituting (3.1) into (2.11).

The solution of (3.2) subject to (2.11) is

$$U_0 = \left\{\begin{array}{l} 1 - erf\left(\dfrac{Z}{2X^{1/2}}\right), \; X \leqslant T, \\ 1 - erf\left(\dfrac{Z}{2T^{1/2}}\right), \; X \geqslant T, \end{array}\right\} \qquad \text{...(3.5)}$$

where

$$erf\, s = \frac{2}{\pi^{1/2}}\int_0^s e^{-y^2}\, dy. \qquad \text{...(3.6)}$$

We see from solution (3.5) that the leading edge signal, which separates the X-independent and T-independent regions, satisfies

$$X = T, \qquad \text{...(3.7)}$$

whose dimensional form has already been displayed as (1.1). Calculation of further terms in the series does not modify the leading edge formula (3.7). In fact the series for U in the region $X > T$ terminates at U_0 because U_n ($n = 1, 2, ...$) satisfies a homogeneous equation and homogeneous boundary conditions. In the region $X < T$ the equation for U_1 is

$$LU_1 = -\frac{1}{2\pi^{1/2}}\frac{Z^2}{X^{3/2}}\exp\left(-\frac{Z^2}{4X}\right). \qquad \text{...(3.8)}$$

The solution to (3.8) that satisfies homogeneous boundary conditions is

$$U_1 = \frac{Z}{4}\left[-\frac{Z}{(\pi X)^{1/2}}\exp\left(-\frac{Z^2}{4X}\right) - 1 + erf\left(\frac{Z}{2X^{1/2}}\right)\right]. \qquad \text{...(3.9)}$$

Away from the surface of the plate

$$1 - erf\left(\frac{Z}{2X^{1/2}}\right) \sim \frac{2}{Z}\left(\frac{X}{\pi}\right)^{1/2}\exp\left(-\frac{Z^2}{4X}\right) \qquad \text{...(3.10)}$$

and

$$\frac{\epsilon U_1}{U_0} \sim -\frac{1}{8}\frac{\epsilon Z^3}{X}\left[1 + O\left(\frac{X}{Z^2}\right)\right] \qquad \text{...(3.11)}$$

$$= O\left(\frac{\omega z^3}{\nu x}\right) \qquad \text{...(3.12)}$$

in dimensional variables.

We notice that the term (3.12) is of order unity if

$$z = O\left(\frac{\nu x}{\omega}\right)^{1/3} \qquad \text{...(3.13)}$$

which means that

$$\left.\begin{aligned} Z &= O(\epsilon^{-1/3}) \\ \epsilon Z &= O(\epsilon^{2/3}) \\ Z^2/X &= O(\epsilon^{-2/3}). \end{aligned}\right\} \qquad \text{...(3.14)}$$

Thus there is a region in which the perturbation term $\epsilon Z\partial U/\partial X$ is much smaller than $\partial U/\partial X$ but the series solution yields successive terms of comparable order. The ratio (3.12) persists when successive terms in the series are calculated. This result suggests that the series solution (and the leading edge signal) have not been correctly formulated.

We can improve the range of validity of the series by using the method of strained coordinates (Lighthill 1949, 1961; Van Dyke 1964). We will see that the solution automatically modifies the formula for the signal.

4. The Strained Coordinate Solution

Define new independent variables χ, ζ, τ by the relations

$$\left.\begin{aligned} X &= \chi f(\epsilon, \zeta) \\ Z &= \zeta \\ T &= \tau \end{aligned}\right\} \qquad \ldots(4.1)$$

where $f(\epsilon, \zeta)$ remains to be determined. We can define appropriate boundary conditions on f as follows. When the mainstream shear is absent, $\epsilon = 0$, and the coordinates x and χ coincide. Consequently

$$f(0, \zeta) = 1. \qquad \ldots(4.2)$$

When $\zeta = 0$ we expect the signal at the plate to be independent of the shear. In other words the new signal formula (which, as we shall see, is given by $\chi = \tau$) should agree with the old signal $X = T$ at $\zeta = 0$. This means that

$$f(\epsilon, 0) = 1. \qquad \ldots(4.3)$$

It follows from (4.2) and (4.3) that for small $\epsilon\zeta$

$$f(\epsilon, \zeta) = 1 + O(\epsilon\zeta). \qquad \ldots(4.4)$$

Derivatives in the old coordinates transform to

$$\left.\begin{aligned} \frac{\partial}{\partial X} &= \frac{1}{f}\frac{\partial}{\partial \chi} \\ \frac{\partial}{\partial Z} &= \frac{\partial}{\partial \zeta} - \chi\frac{f'}{f}\frac{\partial}{\partial \chi} \\ \frac{\partial}{\partial T} &= \frac{\partial}{\partial \tau} \end{aligned}\right\} \qquad \ldots(4.5)$$

where the prime denotes differentiation with respect to ζ.

The governing equation (2.1) transforms under (4.5) to

$$f^2 \mathcal{L} U = [f(f - 1 - \epsilon\zeta) + 2f'^2\chi - ff''\chi]\, U_\chi - 2ff'\chi U_{\chi\zeta} + f'^2\chi^2 U_{\chi\chi} \qquad \ldots(4.6)$$

where the operator

$$\mathcal{L} \equiv \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \chi} - \frac{\partial}{\partial \zeta^2} \qquad \ldots(4.7)$$

and where subscripts denote derivatives. Again we look for a series solution, in the form

$$U(\chi, \zeta, \tau) = U_0(\chi, \zeta, \tau) + \varepsilon U_1(\chi, \zeta, \tau) + \epsilon^2 U_2(\chi, \zeta, \tau) + \dots \qquad \dots(4.8)$$

We obtain the equation for the zeroth order term U_0 by substituting (4.8) into (4.7), setting $\epsilon = 0$ and using (4.2). It comes as no surprise to find that

$$\mathcal{L}\, U_0 = 0. \qquad \dots(4.9)$$

The solution of (4.9), subject to the boundary conditions that correspond to (2.11) in χ, ζ, τ variables, is as we might expect,

$$U_0 = \left\{ \begin{array}{l} 1 - erf\left(\dfrac{\zeta}{2\chi^{1/2}}\right), \chi \leqslant \tau, \\ 1 - erf\left(\dfrac{\zeta}{2\tau^{1/2}}\right), \chi \geqslant \tau. \end{array} \right\} \qquad \dots(4.10)$$

In the region $\chi > \tau$ the series for U terminates at U_0. The leading edge signal, which separates the χ-independent and τ-independent flow regions, is given by

$$\left. \begin{array}{l} \chi = \tau \\ \text{or} \\ x = u_0 t\, f(\epsilon, \zeta). \end{array} \right\} \qquad \dots(4.11)$$

In the region $\chi < \tau$ the equation for the correction term U_1 may be written in the form

$$f^2 \mathcal{L}\, U_1 = \left[F(\epsilon, \zeta)\frac{\zeta}{\chi^{3/2}} + G(\epsilon, \zeta)\frac{2}{\chi^{1/2}}\right]\frac{1}{8\pi^{1/2}} \exp\left(-\frac{\zeta^2}{4\chi}\right) \qquad \dots(4.12)$$

where

$$F(\epsilon, \zeta) = \zeta^2 f'^2 + 4\zeta f' f + 4f(f - 1 - \epsilon\zeta) \qquad \dots(4.13)$$

and

$$G(\epsilon, \zeta) = f'^2\zeta - 4ff' - 2ff''\zeta. \qquad \dots(4.14)$$

Away from the surface of the plate

$$U_1 = \left\{c\frac{\zeta^2}{\chi^{1/2}} F\left[1 + O\left(\frac{\chi}{\zeta^2}\right)\right] + O\left(\zeta\chi^{1/2}G\right)\right\} \exp\left(-\frac{\zeta^2}{4\chi}\right) \qquad \dots(4.15)$$

where c is a constant. From (3.10), (4.10) and (4.15) we see that

$$\frac{\epsilon U_1}{U_0} = a_0\frac{\zeta^2}{\chi} F + O(G) \qquad \dots(4.16)$$

where a_0 is a constant. In view of (4.4)

$$\left. \begin{array}{l} F = O(\epsilon\zeta) \\ G = O(\epsilon\zeta). \end{array} \right\} \qquad \dots(4.17)$$

Compare the two results (3.11) and (4.16). The term $\epsilon U_1/U_0$ on the left-hand side of (4.16) will be of order unity in the region given by (3.13) unless the term involving F is annulled by a suitable choice of f. We can show, by continuing the series for U, that

$$\frac{\epsilon U_{n+1}}{U_n} = a_n F \frac{\zeta^2}{\chi} + O(G), \qquad \text{...(4.18)}$$

where a_n is a constant. It follows that by choosing f to make F identically zero we can guarantee that $\frac{\epsilon U_{n+1}}{U_n} = O(\epsilon\zeta)$ for all n, in the region where $\zeta = O(\epsilon^{-1/3})$. The solution for f then determines the modified form of the leading edge signal (4.11).

5. The Leading Edge Signal

When $F \equiv 0$ eqn. (4.13) shows that

$$\zeta^2 f'^2 + 4\zeta f' f + 4f(f - 1 - \epsilon\zeta) = 0. \qquad \text{...(5.1)}$$

Equation (5.1) can be thought of as a quadratic in $\zeta f'$. Its roots are

$$\zeta f' = -2f \pm 2(1 + \epsilon\zeta)^{1/2} f^{1/2}. \qquad \text{...(5.2)}$$

We may express (5.2) as a linear equation in $f^{1/2}$,

$$(f^{1/2})' + \frac{1}{\zeta} f^{1/2} = \pm \frac{1}{\zeta}(1 + \epsilon\zeta)^{1/2}. \qquad \text{...(5.3)}$$

The solution of (5.3) that satisfies boundary condition (4.3) is

$$f = \frac{4}{9}\left[\frac{(1 + \epsilon\zeta)^{3/2} - 1}{\epsilon\zeta}\right]^2. \qquad \text{...(5.4)}$$

By substituting (5.4) into (4.11) we find that the equation for the leading edge signal now reads

$$x = u_0 t \, \frac{4}{9}\left[\frac{(1 + \omega z/U_0)^{3/2} - 1}{\omega z/U_0}\right]^2. \qquad \text{...(5.5)}$$

For $\omega z/u_0 \ll 1$ formula (5.5) expands to the form

$$x = u_0 t\left[1 + \frac{1}{2}\left(\frac{\omega z}{U_0}\right) + O\left(\frac{\omega z}{U_0}\right)^2\right]. \qquad \text{...(5.6)}$$

6. Conclusion

The main result of the analysis is found in equation (5.6). This equation represents, at a fixed time t, a curve in the x - z plane that divides the x-independent region from the t-independent region.

For positive z the second term on the right-hand side of (5.6) is positive. Consequently, the effect of the external shear is to sweep the signal 'downstream' relative to the signal position $x = u_0 t$ at the plate $z = 0$. This is what we would expect. However, according to (5.6) the signal is not convected with the mainstream speed $u_0 + \omega z$. If it were, the signal position would have been given by

$$x = u_0 t \left(1 + \frac{\omega z}{u_0} \right) \qquad \ldots(6.1)$$

In other words a fluid particle that at $t = 0$ lay upstream of the plate would, if convected with the mainstream speed, draw near to and eventually overtake the signal position at the same station z, losing x-dependence and gaining t-dependence as it did so.

We may extend the analysis to below the plate, where $z < 0$. Equation (5.6) shows that the signal is swept 'upstream' relative to the signal at the plate. A fluid particle, if travelling with mainstream speed in a fixed station $z < 0$, would be overtaken by the signal at that station, losing its t-dependence and gaining x-dependence in the process.

In view of these puzzling results we ought perhaps to treat formula (5.6) with caution. It must be viewed in the light of the assumptions underlying the analysis. These included a constant pressure, an unbounded shear flow, weak vorticity interaction and the Oseen linearization applied to the boundary layer equation.

References

Ferri, A., and Libby, P. A. (1954). Note on an interaction between the boundary layer and the inviscid flow. *J. aeronaut. Sci.*, **21**, 130.

Glauert, M. B. (1957). The boundary layer in simple shear past a flat plate. *J. aeronaut. Sci.*, **24**, 848.

——— (1962). The pressure gradient induced by shear flow past a flat plate. *J. aeronaut. Sci.*, **29**, 540.

Li, T. Y. (1955). Simple shear past a flat plate in an incompressible fluid of small viscosity. *J. aeronaut. Sci.*, **22**, 651.

——— (1956). Effects of free-stream vorticity on the behaviour of a viscous boundary layer. *J. aeronaut. Sci.*, **23**, 1128.

Lighthill, M. J. (1949). A technique for rendering approximate solutions to physical problems uniformly valid. *Phil. Mag.* (*Series* 7), **40**, 1179.

——— (1961). A technique for rendering approximate solutions to physical problems uniformly valid. *Z. Flugwiss*, **9**, 267.

Mark, R. M. (1962). On shear flows past flat plates. *J. Fluid Mech.*, **14**, 452.

——— (1966). On the theory of asymmetric shear flows past flat plates. *J. Fluid Mech.*, **25**, 457.

Murray, J. D. (1961). The boundary layer on a flat plate in a stream with uniform shear. *J. Fluid Mech.*, **11**, 309.

Stewartson, K. (1951). On the implusive motion of a flat plate in a viscous fluid. *Q. Jl Mech. appl. Math.*, **4**, 182.

Stewartson, K. (1960). The theory of unsteady laminar boundary layers. *Adv. appl. Mech.*, **6**, 1.

Toomre, A.. and Rott, N. (1964). On the pressure induced by the boundary layer on a flat plate in shear flow. *J. Fluid Mech.*, **19**, 1.

Van Dyke, M. (1964). Perturbation Methods in Fluid Mechanics. Academic Press, New York.