A PLANE-SYMMETRIC NULL ELECTROMAGNETIC FIELD

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The necessary conditions for the cylindrically-symmetric metric of Marder (1958) to represent a null electromagnetic field have been obtained in terms of the curvature components. Considering a special case of the Marder metric a plane-symmetric null electromagnetic field has been derived and some properties of the model have been studied.

1. INTRODUCTION

The general cylindrically-symmetric metric of Marder (1958) is considered in the form,

$$ds^2 = A^2(dt^2 - dx^2) - B^2dy^2 - C^2dz^2 \qquad ...(1.1)$$

where A, B, C are functions of x and t alone. The electromagnetic energy-momentum tensor, E_i^j, is given by,

$$E_i^j = -F_{ik}F^{jk} + \tfrac{1}{4} g_i^j F_{lm}F^{lm} \qquad ...(1.2)$$

where F_{ij} is the electromagnetic field tensor which satisfies the Maxwell equations if,

$$F_{ij} = P_{i,j} - P_{j,i} \qquad ...(1.3)$$

and

$$(\sqrt{-g}\ F^{ij})_{,j} = 0. \qquad ...(1.4)$$

P_i being the four-potential and a comma stands for partial derivation.

The eight algebraically independent and non-vanishing components of the curvature tensor, R_{hijk}, for the metric (1.1) are given by (Singh and Abdussattar 1973)

$$R_{1212} \equiv f_1 = B\left\{B_{11} - \frac{A_1B_1 + A_4B_4}{A}\right\}$$

$$R_{1313} \equiv \bar{f}_1 = C\left\{C_{11} - \frac{A_1C_1 + A_4C_4}{A}\right\}$$

(*equations continued on p.* 1226)

$$\left.\begin{aligned}
R_{1224} &\equiv f_2 = B\left\{\frac{A_1B_4 + A_4B_1}{A} - B_{14}\right\}\\
R_{1334} &\equiv \bar{f}_2 = C\left\{\frac{A_1C_4 + A_4C_1}{A} - C_{14}\right\}\\
R_{2424} &\equiv f_3 = B\left\{B_{44} - \frac{A_1B_1 + A_4B_4}{A}\right\}\\
R_{3434} &\equiv \bar{f}_3 = C\left\{C_{44} - \frac{A_1C_1 + A_4C_4}{A}\right\}\\
R_{1414} &\equiv f_4 = \left\{(A_1^2 - A_4^2) + A\,(A_{44} - A_{11})\right\}\\
R_{2323} &\equiv f_5 = \frac{BC}{A^2}\left\{B_1C_1 - B_4C_4\right\}.
\end{aligned}\right\} \quad \ldots(1.5)$$

The suffixes 1 and 4 after the symbols A, B, C denote ordinary partial differentiation with respect to x and t respectively. The field equations

$$R_i^j - \tfrac{1}{2}Rg_i^j = -\,8\pi E_i^j \qquad \ldots(1.6)$$

for the regions of the space containing electromagnetic field but no matter are given by

$$\left.\begin{aligned}
&\frac{f_5}{B^2C^2} - \frac{f_3}{A^2B^2} - \frac{\bar{f}_3}{A^2C^2} = -\,8\pi E_1^1\\
&\frac{\bar{f}_1 - \bar{f}_3}{A^2C^2} - \frac{f_4}{A^4} = -\,8\pi E_2^2\\
&\frac{f_1 - f_3}{A^2B^2} - \frac{f_4}{A^4} = -\,8\pi E_3^3\\
&\frac{f_5}{B^2C^2} + \frac{f_1}{A^2B^2} + \frac{\bar{f}_1}{A^2C^2} = -\,8\pi E_4^4\\
&\frac{f_2}{A^2B^2} + \frac{\bar{f}_2}{A^2C^2} = -\,8\pi E_4^1 = 8\pi E_1^4.
\end{aligned}\right\} \quad \ldots(1.7)$$

Considering the eigenvalue equation

$$\left| E_i^j - \lambda g_i^j \right| = 0. \qquad \ldots(1.8)$$

the four eigenvalues of the electromagnetic tensor are obtained as

$$\left.\begin{aligned}\lambda_1 &= \frac{E^1_1+E^4_4}{2}+\tfrac{1}{2}\sqrt{\{(E^1_1-E^4_4)^2+4E^4_1E^1_4\}}\\ \lambda_2 &= E^2_2\\ \lambda_3 &= E^3_3\\ \text{and}\qquad \lambda_4 &= \frac{E^1_1+E^4_4}{2}-\tfrac{1}{2}\sqrt{\{(E^1_1-E^4_4)^2+4E^4_1E^1_4\}}.\end{aligned}\right\}\quad\ldots(1.9)$$

For a null electromagnetic field we must have

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=0. \qquad \ldots(1.10)$$

This gives

$$\left.\begin{aligned}E^2_2 &= E^3_3 = 0\\ E^1_1+E^4_4 &= 0\\ \text{and}\qquad E^1_1E^4_4 &= E^1_4E^4_1.\end{aligned}\right\}\quad\ldots(1.11)$$

Making use of the field equations (1.7), these conditions can be expressed in terms of the curvature components as follows:

$$\left.\begin{aligned}&\frac{f_1-f_3}{B^2}=\frac{\bar{f}_1-\bar{f}_3}{C^2}=\frac{f_4}{A^2}\\ &\frac{f_5}{C^2}+\frac{f_1-f_3}{A^2}=0\\ \text{and}\qquad &\left\{\frac{f_2}{B^2}+\frac{\bar{f}_2}{C^2}\right\}^2=\left\{\frac{f_1}{B^2}+\frac{\bar{f}_3}{C^2}\right\}^2.\end{aligned}\right\}\quad\ldots(1.12)$$

2. A Special Case

In order to obtain a specific solution representing a null electromagnetic field the line-element (1.1) is considered in the form

$$ds^2=A^2(dt^2-dx^2)-B^2(dy^2+dz^2) \qquad \ldots(2.1)$$

where A is a function of x and t and B is a function of t alone. Conditions (1.12) in this case are,

$$\left\{\frac{A_4}{A}\right\}_4-\left\{\frac{A_1}{A}\right\}_1=-\frac{B_{44}}{B}, \qquad \ldots(2.2)$$

$$\left\{ BB_4 \right\}_4 = 0 \qquad \text{...(2.3)}$$

and

$$\left\{ \frac{A_4}{A} \right\} + \left\{ \frac{A_1}{A} \right\} = \frac{B_{44}}{2B_4}. \qquad \text{...(2.4)}$$

Equation (2.3) gives,

$$B = \left\{ \alpha_0 t + \beta_0 \right\}^{\frac{1}{2}} \qquad \text{...(2.5)}$$

where α_0 and β_0 are constants of integration. Using (2.5) in eqns. (2.2) and (2.4) we get,

$$\left\{ \frac{A_4}{A} \right\}_4 - \left\{ \frac{A_1}{A} \right\}_1 = \frac{\alpha_0^2}{4(\alpha_0 t + \beta_0)^2} \qquad \text{...(2.6)}$$

and

$$\left\{ \frac{A_4}{A} \right\} + \left\{ \frac{A_1}{A} \right\} = -\frac{\alpha_0}{4(\alpha_0 t + \beta_0)}. \qquad \text{...(2.7)}$$

A solution of eqns. (2.6) and (2.7) is obtained as

$$A = \gamma_0 \left\{ \alpha_0 t + \beta_0 \right\}^{-1/4} \exp \left\{ \frac{t^2}{2} + \frac{x^2}{2} - tx \right\} \qquad \text{...(2.8)}$$

where γ_0 is another constant of integration. Consequently the line-element (2.1) can be written as,

$$ds^2 = \left\{ \gamma_0 (\alpha_0 t + \beta_0)^{-1/4} \exp \left(\frac{t^2}{2} + \frac{x^2}{2} - tx \right) \right\}^2 (dt^2 - dx^2)$$
$$- \left\{ (\alpha_0 t + \beta_0)^{1/2} \right\}^2 (dy^2 + dz^2). \qquad \text{...(2.9)}$$

With the help of suitable transformations the line-element (2.9) can be more conveniently expressed as

$$ds^2 = \frac{\exp (t^2 + x^2 - 2tx)}{\sqrt{1 + \alpha t}} (dt^2 - dx^2) - \{1 + \alpha t\} (dy^2 + dz^2) \qquad \text{...(2.10)}$$

where α is a positive arbitrary constant.

3. Some Properties of the Field

The non-vanishing components of the electromagnetic energy tensor, E_i^j, for the metric (2.10) are as follows:

$$- 8\pi E^1_1 = 8\pi E^4_4 = 8\pi E^1_4 = - 8\pi E^4_1$$
$$= \frac{\alpha(t-x)}{\sqrt{1+\alpha t}\ \exp\,(t^2+x^2-2tx)}. \qquad ...(3.1)$$

It follows that there is a null eigenvector, K^i, associated with E^{ij} ($= \sigma K^i K^j$) given by

$$K^i = \left\{\frac{\sqrt{\alpha(t-x)}}{\sqrt{8\pi\sigma}\ \exp\,(t^2+x^2-2tx)}, 0, 0, \frac{\sqrt{\alpha(t-x)}}{\sqrt{8\pi\sigma}\ \exp\,(t^2+x^2-2tx)}\right\}. \qquad ...(3.2)$$

For the values of E^j_i given by (3.1), the set of eqns. (1.2) yield the solution

$$\left.\begin{aligned} F_{12} &= F_{24} \\ F_{13} &= F_{34} \\ F_{14} &= F_{23} = 0 \end{aligned}\right\} \qquad ...(3.3)$$

and

$$\{F_{12}\}^2 + \{F_{13}\}^2 = \frac{\alpha(t-x)}{8\pi}. \qquad ...(3.4)$$

We take P_i as

$$P_i = \left\{0, \frac{\phi(t-x)}{\sqrt{8\pi}}, \frac{\psi(t-x)}{\sqrt{8\pi}}, 0\right\}. \qquad ...(3.5)$$

This gives

$$\left.\begin{aligned} F_{12} &= F_{24} = \frac{\phi'}{\sqrt{8\pi}} \\ F_{13} &= F_{34} = \frac{\psi'}{\sqrt{8\pi}} \end{aligned}\right\} \qquad ...(3.6)$$

and

$$F_{14} = F_{23} = 0$$

where a dash denotes differentiation with respect to $u \equiv (t-x)$. We see that for P_i given by (3.5), the set of eqns. (3.3) are satisfied and eqn. (3.4) demands that

$$\{\phi'\}^2 + \{\psi'\}^2 = \alpha(t-x). \qquad ...(3.7)$$

Also for the values of F_{ij} given by eqns. (3.6), the set of eqns. (1.4) are satisfied. In order that the energy-density of the field be positive definite we must have

$$R^4_4 < 0. \qquad ...(3.8)$$

This gives

$$\frac{\alpha(t-x)}{\sqrt{1+\alpha t}\,\exp(t^2+x^2-2tx)} > 0. \qquad ...(3.9)$$

Condition (3.7) and (3.9) are fulfilled provided we take $\alpha > 0$ and $t > x$.

The surviving components of the Weyl conformal curvature tensor, C_{hijk}, for the model (2.10) are as follows:

$$C^{12}_{12} = C^{13}_{13} = C^{24}_{24} = C^{34}_{34} = -\tfrac{1}{2}C^{14}_{14} = -\tfrac{1}{2}C^{23}_{23}$$

$$= \frac{\alpha^2}{8(1+\alpha t)^{3/2}\exp(t^2+x^2-2tx)}. \qquad ...(3.10)$$

Out of the fourteen scalar invariants of second order (Narlikar and Karmarkar 1949) evaluated for the model (2.10) only two are non-zero; namely

$$J_1 = \frac{3}{4}\cdot\frac{\alpha^4}{(1+\alpha t)^3}\cdot\frac{1}{\exp(2t^2+2x^2-4tx)} \qquad ...(3.11)$$

and

$$J_2 = -\frac{3}{16}\cdot\frac{\alpha^6}{(1+\alpha t)^{9/2}}\cdot\frac{1}{\exp(3t^2+3x^2-6tx)}. \qquad ...(3.12)$$

Petrov-Pirani classification of the model reveals that it is of Petrov type I degenerate. This shows that an electromagnetically radiative field need not necessarily be gravitationally radiative also.

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