ON FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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Let R denote the class of normalized functions f regular in $E = \{z : |z| < 1\}$ such that Re $f'(z) > 0$ in E and S^* be the class of starlike univalent functions. Let B_k and U_k be the classes of regular functions which generalize the classes R and S^* respectively in the same manner as the class V_k of functions with boundary rotation bounded by $k\pi$ generalizes the class K of convex functions. Here several properties of the classes B_k and U_k are investigated.

1. INTRODUCTION

Let V_k denote the class of holomorphic functions f defined in $E = \{z : |z| < 1\}$ given by

$$f(z) = z + a_2 z^2 + \dots \qquad \dots(1.1)$$

and which maps E conformally onto a image domain of boundary rotation atmost $k\pi$. Paatero (1933) has shown that $f(z) \in V_k$ if and only if

$$f'(z) = \exp\left\{-\int_0^{2\pi} \log(1 - ze^{-it})\, dm(t)\right\} \qquad \dots(1.2)$$

where $m(t) \in M_k$, i.e. $m(t)$ is a real-valued function of bounded variation on $[0, 2\pi]$ satisfying the conditions,

$$\int_0^{2\pi} dm(t) = 2, \quad \int_0^{2\pi} |dm(t)| \leqslant k. \qquad \dots(1.3)$$

It is easy to see that V_2 is the class of convex univalent functions.

Let U_k denote the class of holomorphic functions $f(z)$ of the form (1.1) having the representation

$$f(z) = z\exp\left\{-\int_0^{2\pi} \log(1 - ze^{-it})\, dm(t)\right\} \qquad \dots(1.4)$$

for some $m(t) \in M_k$. Clearly U_2 coincides with class of starlike univalent functions.

Let P_k denote the class of regular functions $p(z)$ in E such that $p(0) = 1$ and having the representation

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \, dm(t) \qquad \ldots(1.5)$$

where $m(t) \in M_k$. Let B_k denote the class of regular functions $f(z)$ in E of the form (1.1) such that $f'(z)$ is in P_k. The classes U_k and P_k were introduced by Pinchuk (1971).

It is the object of this paper to determine the radius of convexity for the classes U_k and B_k and in the sequel to investigate some other properties of the above mentioned classes.

2. Radii of Convexity

Lemma 1 — Suppose $p(z) \in P_k$, then

$$\operatorname{Re}\left\{z \frac{p'(z)}{p(z)}\right\} \geqslant \frac{-r(k - 4r + kr^2)}{(1 - r^2)(1 - kr + r^2)}, \text{ where } |z| = r, k \geqslant 4 \qquad \ldots(2.1)$$

and $\quad |z| < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$. For $2 \leqslant k \leqslant 4$

$$\operatorname{Re}\left\{z \frac{p'(z)}{p(z)}\right\} \geqslant \frac{-2kr + (8 - 4k + k^2) r^2 - 2kr^3}{2(1 - r^2)(1 - kr + r^2)}. \qquad \ldots(2.2)$$

The above inequality is sharp for the function $p(z) = \dfrac{1 - kz + z^2}{1 - z^2}$ when $k \geqslant 4$.

Proof : Suppose $p(z) \in P_k$. Then there exists a function $f(z) \in V_k$ (Pinchuk 1971) such that

$$p(z) = 1 + \frac{zf''(z)}{f'(z)}. \qquad \ldots(2.3)$$

Also it is known that (Moulis 1972) when $f(z) \in V_k$

$$|\{f, z\}| = \left| \left\{\frac{f''(z)}{f'(z)}\right\}' - \frac{1}{2}\left\{\frac{f''(z)}{f'(z)}\right\}^2 \right| \leqslant \begin{cases} \dfrac{k^2 - 4}{2(1 - |z|^2)^2}, & k \geqslant 4; \\[2ex] \dfrac{2(k - 1)}{(1 - |z|^2)^2}, & 2 \leqslant k \leqslant 4. \end{cases} \qquad \ldots(2.4)$$

First let us consider the case $k \geqslant 4$. From (2.3) and (2.4) we get

$$|zp'(z) + \tfrac{1}{2}(1 - (p(z))^2)| \leqslant \frac{(k^2 - 4)|z|^2}{2(1 - |z|^2)^2}.$$

Since $\operatorname{Re} p(z) > 0$ for $|z| < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$, (Pinchuk 1971) $p(z) \neq 0$ for $|z| < R_0$, for $|z| < R_0$ we have

$$\left| z \frac{p'(z)}{p(z)} + \tfrac{1}{2}\left(\frac{1}{p(z)} - p(z)\right)\right| \leqslant \frac{(k^2 - 4)\,|z|^2}{2(1 - |z|^2)^2\,|p(z)|}$$

or

$$\operatorname{Re}\left\{z \frac{p'(z)}{p(z)}\right\} \geqslant \frac{\operatorname{Re} p(z)}{2} - \operatorname{Re}\left\{\frac{1}{2p(z)}\right\} - \frac{(k^2 - 4)\,|z|^2}{2(1 - |z|^2)^2\,|p(z)|}$$

$$\geqslant \operatorname{Re}\frac{p(z)}{2} - \frac{1}{2\operatorname{Re} p(z)} - \frac{(k^2 - 4)\,|z|^2}{2(1 - |z|^2\operatorname{Re} p(z)} \qquad \text{...(2.5)}$$

where we have used the fact that $\operatorname{Re}\dfrac{1}{p(z)} \leqslant \dfrac{1}{\operatorname{Re} p(z)}$ when $\operatorname{Re} p(z) > 0$. Also since $p(z) \in P_k$ we have (Pinchuk 1971)

$$\frac{1 - kr + r^2}{1 - r^2} \leqslant \operatorname{Re} p(z) \leqslant \frac{1 + kr + r^2}{1 - r^2} \text{ where } |z| = r. \qquad \text{...(2.6)}$$

From (2.5) and (2.6) we get

$$\operatorname{Re}\left\{z \frac{p'(z)}{p(z)}\right\} \geqslant \tfrac{1}{2}\,\frac{1 - kr + r^2}{1 - r^2} - \tfrac{1}{2}\,\frac{1 - r^2}{1 - kr + r^2} - \frac{(k^2 - 4)\,r^2}{2(1 - r^2)(1 - kr + r^2)}$$

$$= \frac{-\,r(k - 4r + kr^2)}{(1 - r^2)(1 - kr + r^2)}$$

where $|z| = r < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$. This is sharp for the function

$$p(z) = \frac{1 - kz + z^2}{1 - z^2}.$$

For $2 \leqslant k \leqslant 4$ using the other estimate in (2.4) we get the inequality (2.2). The proof of the lemma is complete.

Theorem 1 — Let $f(z) \in B_k$, $k \geqslant 4$; then it maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of the equation

$$r^4 - 4r^2 + 2kr - 1 = 0. \qquad \text{...(2.7)}$$

The above bound r_0 is sharp. When $2 \leqslant k \leqslant 4$, $f(z)$ maps $|z| < r_1$ onto a convex domain where r_1 is the least positive root of the equation

$$2r^4 - (8 - 4k + k^2)\,r^2 + 4kr - 2 = 0. \qquad \text{...(2.8)}$$

However, the bound r_1 is not sharp when $2 \leqslant k < 4$.

PROOF : Since $f(z) \in B_k$ there exists a $p(z) \in P_k$ such that

$f'(z) = p(z)$.

Hence

$$\operatorname{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} = \operatorname{Re}\left\{1 + z\,\frac{p'(z)}{p(z)}\right\}. \quad \text{...(2.9)}$$

For $k \geqslant 4$ using (2.1) in the Lemma 1 we have

$$\operatorname{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} \geqslant 1 - \frac{(k - 4r + kr^2)\,r}{(1 - r^2)(1 - kr + r^2)},$$

for $|z| = r < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$.

Thus

$$\operatorname{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} \geqslant \frac{1 - 2kr + 4r^2 - r^4}{(1 - r^2)(1 - kr + r^2)} > 0$$

provided $r^4 - 4r^2 + 2kr - 1 < 0$.

Let $Q(r) = r^4 - 4r^2 + 2kr - 1$. Then $Q(r) = 0$ has a unique root in $(0, R_0)$ where $R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$. This proves the first part of the theorem. For $2 \leqslant k \leqslant 4$, using (2.9) and (2.2) we will get

$$\operatorname{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} \geqslant 1 + \frac{-2kr + (8 - 4k + k^2)\,r^2 - 2kr^3}{2(1 - r^2)(1 - kr + r^2)}$$

for $|z| = r < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$.

Hence $\operatorname{Re}\left\{1 + z\,\dfrac{f''(z)}{f'(z)}\right\} > 0$ for $|z| = r < r_1$ where r_1 is the least positive root of eqn. (2.8).

Consider the function $f(z)$ defined by $f'(z) = \dfrac{1 - kz + z^2}{1 - z^2}$.

$$\text{Then } 1 + z\,\frac{f''(z)}{f'(z)} = \frac{1 - 2kz + 4z^2 - z^4}{(1 - z^2)(1 - kz + z^2)}$$

$$= 0 \quad \text{by (2.7) for } z = r_0.$$

This shows that the bound r_0 is sharp when $k \geqslant 4$.

Corollary 1 — It follows from Theorem 1, that each function $f \in B_2$ maps $|z| < R$ onto a convex domain where R is the least positive root of the equation

$r^4 - 2r^2 + 4r - 1 = 0$. R is however less than the sharp estimate $(\sqrt{2} - 1)$ for the radius of convexity obtained for this class by MacGregor (1962).

Theorem 2 — Suppose $f(z) \in U_k$. Then $f(z)$ maps $|z| < r_2$ onto a convex domain where r_2 is the least positive root of the equation

$$1 - 3kr + (k^2 + 6)\, r^2 - 3kr^3 + r^4 = 0 \qquad \ldots(2.10)$$

where $|z| = r$ and $k \geqslant 4$. The bound r_2 is sharp. For $2 \leqslant k \leqslant 4$, then $f(z)$ maps $|z| < r_3$ onto a convex domain where r_3 is the least positive root of the equation

$$2 - 6kr + (12 - 4k + 3k^2)\, r^2 - 4kr^3 + 2r^4 = 0. \qquad \ldots(2.11)$$

However the bound r_3 is not sharp when $2 \leqslant k < 4$.

PROOF : Since $f(z) \in U_k$ there exists a $p(z) \in P_k$ such that (Pinchuk 1971)

$$z\,\frac{f'(z)}{f(z)} = p(z).$$

Hence

$$\text{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} = \text{Re}\left\{z\,\frac{p'(z)}{p(z)}\right\} + \text{Re}\left\{z\,\frac{f'(z)}{f(z)}\right\}.$$

Using (2.1) and (2.6) we get for $k \geqslant 4$ and $|z| = r$,

$$\text{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} \geqslant \frac{-\,r(k - 4r + kr^2)}{(1 - r^2)(1 - kr + r^2)} + \frac{1 - kr + r^2}{1 - r^2}$$

$$\geqslant \frac{(1 - kr + r^2)^2 - r\,(k - 4r + kr^2)}{(1 - r^2)(1 - kr + r^2)}$$

where $|z| = r < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$.

Hence $\text{Re}\left\{1 + z\,\dfrac{f''(z)}{f'(z)}\right\} \geqslant 0$ provided $Q(r) = 1 - 3kr + (k^2 + 6)\, r^2 - 3kr^3 + r^4 > 0$.

The equation $Q(r) = 0$ has a unique positive root in $(0, R_0)$.

Hence the result. For the function $f(z) = \dfrac{z(1 - z)^{(k/2)-1}}{(1 + z)^{(k/2)+1}} \in U_k$ we have

$$z\,\frac{f'(z)}{f(z)} = \frac{1 - kz + z^2}{1 - z^2}.$$

Hence $1 + z\,\dfrac{f''(z)}{f'(z)} = \dfrac{-\,kz + 4z^2 - kz^3}{(1 - z^2)(1 - kz + z^2)} + \dfrac{1 - kz + z^2}{(1 - z^2)}$,

$$\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} = \frac{(1 - kz + z^2)^2 - z(k - 4z + kz^2)}{(1 - z^2)(1 - kz + z^2)}$$

$$= \frac{1 - 3kz + (k^2 + 6)\,z^2 - 3kz^3 + z^4}{(1 - z^2)(1 - kz + z^2)} = 0$$

for $z = r_2$ by (2.10). Hence the result is sharp.

For $2 \leqslant k \leqslant 4$, using (2.2) and (2.6) we have

$$\text{Re}\left\{1 + z\,\frac{f''(z)}{f'(z)}\right\} \geqslant \frac{-2kr + (8 - 4k + k^2)\,r^2 - 2kr^3}{2(1 - r^2)(1 - kr + r^2)} + \frac{1 - kr + r^2}{1 - r^2}$$

where $| z | = r < R_0 = \dfrac{k - \sqrt{k^2 - 4}}{2}$.

Hence $\text{Re}\left\{1 + z\,\dfrac{f''(z)}{f'(z)}\right\} > 0$ provided

$$T(r) = 2 - 6kr + (12 - 4k + 3k^2)\,r^2 - 6kr^3 + 2r^4 > 0.$$

Also $T(r) = 0$ has a root in $(0, R_0)$. Hence the result follows.

Corollary 2 — It follows from Theorem 2 that each function $f \in U_2$ maps $| z | < R$ onto a convex domain, where R is the least positive root of $r^3 - 3r^2 + 5r - 1 = 0$. R is however less than the well-known sharp estimate $(2 - \sqrt{3})$ for the radius of convexity of this class.

3 Other Properties of the above Mentioned Classes

Theorem 3 — Let $f(z) = 1 + \sum_{1}^{\infty} b_\nu z^\nu$ and $g(z) = 1 + \sum_{1}^{\infty} c_\nu z^\nu$ belong to P_k.

Then $f(z) * g(z) = 1 + \frac{1}{2} \sum_{1}^{\infty} b_\nu c_\nu z^\nu$ belongs to P_K where $K = \dfrac{k^2}{2}$.

Proof: Let $f(z)$ and $g(z)$ belong to P_k then

$$f(z) = 1 + \sum_{1}^{\infty} b_\nu z^\nu = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)$$

$$g(z) = 1 + \sum_{1}^{\infty} b_\nu z^\nu = \left(\frac{k}{4} + \frac{1}{2}\right) p_3(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_4(z)$$

where $p_1(z)$, $p_2(z)$, $p_3(z)$ and $p_4(z)$ all belong to P_2.

Let $p_i(z) = 1 + \sum_{1}^{\infty} p_{i\nu} z^\nu$, $i = 1, 2, 3$ and 4.

Let $f(z) * g(z) = 1 + \sum_{1}^{\infty} \frac{b_\nu c_\nu}{2} z^\nu$.

Here
$$b_\nu = \left(\frac{k}{4} + \tfrac{1}{2}\right) p_{1\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right) p_{2\nu}$$
$$c_\nu = \left(\frac{k}{4} + \tfrac{1}{2}\right) p_{3\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right) p_{4\nu}.$$

Hence
$$\begin{aligned}\frac{b_\nu c_\nu}{2} &= \tfrac{1}{2}\left[\left(\frac{k}{4} + \tfrac{1}{2}\right) p_{1\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right) p_{2\nu}\right] \\ &\quad \times \left[\left(\frac{k}{4} + \tfrac{1}{2}\right) p_{3\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right) p_{4\nu}\right] \\ &= \left\{\left(\frac{k}{4} + \tfrac{1}{2}\right)^2 \frac{p_{1\nu} p_{3\nu}}{2} - \left(\frac{k}{4} - \tfrac{1}{2}\right)\left(\frac{k}{4} + \tfrac{1}{2}\right) \frac{p_{2\nu} p_{3\nu}}{2}\right. \\ &\quad \left. - \left(\frac{k}{4} + \tfrac{1}{2}\right)\left(\frac{k}{4} - \tfrac{1}{2}\right) \frac{p_{1\nu} p_{4\nu}}{2} + \left(\frac{k}{4} - \tfrac{1}{2}\right)^2 \frac{p_{2\nu} p_{4\nu}}{2}\right\} \\ &= \left\{\left(\frac{k}{4} + \tfrac{1}{2}\right)^2 q_{1\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right)\left(\frac{k}{4} + \tfrac{1}{2}\right) q_{2\nu}\right. \\ &\quad \left. - \left(\frac{k}{4} + \tfrac{1}{2}\right)\left(\frac{k}{4} - \tfrac{1}{2}\right) q_{3\nu} + \left(\frac{k}{4} - \tfrac{1}{2}\right)^2 q_{4\nu}\right\}, \text{ say}\end{aligned}$$

Then

$$\begin{aligned}\tfrac{1}{2} b_\nu c_\nu &= \left(\frac{k}{4} + \tfrac{1}{2}\right)\left\{\left(\frac{k}{4} + \tfrac{1}{2}\right) q_{1\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right) q_{2\nu}\right\} \\ &\quad - \left(\frac{k}{4} - \tfrac{1}{2}\right)\left\{\left(\frac{k}{4} + \tfrac{1}{2}\right) q_{3\nu} - \left(\frac{k}{4} - \tfrac{1}{2}\right) q_{4\nu}\right\}.\end{aligned}$$

Hence $1 + \tfrac{1}{2} \sum_{1}^{\infty} b_\nu c_\nu z^\nu = \left(\frac{k}{4} + \tfrac{1}{2}\right) Q_1(z) - \left(\frac{k}{4} - \tfrac{1}{2}\right) Q_2(z)$

where

$$Q_1(z) = \left(\frac{k}{4} + \tfrac{1}{2}\right) \sum_{1}^{\infty} q_{1\nu} z^\nu - \left(\frac{k}{4} - \tfrac{1}{2}\right) \sum_{1}^{\infty} q_{2\nu} z^\nu,$$

$$Q_2(z) = \left(\frac{k}{4} + \tfrac{1}{2}\right) \sum_{1}^{\infty} q_{3\nu} z^\nu - \left(\frac{k}{4} - \tfrac{1}{2}\right) \sum_{1}^{\infty} q_{4\nu} z^\nu.$$

Also $1 + \sum_{\nu=1}^{\infty} q_{i\nu} z^{\nu} \in P_2$ for $i = 1, 2, 3, 4$ (Nehari and Netanyahu 1957).

Hence $Q_1(z)$ and $Q_2(z) \in P_k$. Using (1.5) we get

$$Q_i(z) = \tfrac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \, dm(t), \text{ where } m(t) \in M_k \text{ for } i = 1, 2.$$

Therefore

$$\left(\frac{k}{4} + \tfrac{1}{2}\right) Q_1(z) - \left(\frac{k}{4} - \tfrac{1}{2}\right) Q_2(z)$$

$$= \tfrac{1}{2} \int_0^{2\pi} \left(\frac{1 + ze^{-it}}{1 - ze^{-it}}\right) \left[\left(\frac{k}{4} + \tfrac{1}{2}\right) dm_1(t) - \left(\frac{k}{4} - \tfrac{1}{2}\right) dm_2(t)\right]$$

$$= \tfrac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \, d\mu(t) \text{ (say)}$$

where

$$d\mu(t) = \left(\frac{k}{4} + \tfrac{1}{2}\right) dm_1(t) - \left(\frac{k}{4} - \tfrac{1}{2}\right) dm_2(t).$$

Then

$$\int_0^{2\pi} d\mu(t) = \left(\frac{k}{4} + \tfrac{1}{2}\right) 2 - \left(\frac{k}{4} - \tfrac{1}{2}\right) 2 = 2$$

and

$$\int_0^{2\pi} | \, d\mu(t) \, | \leqslant \left(\frac{k}{4} + \tfrac{1}{2}\right) k + \left(\frac{k}{4} - \tfrac{1}{2}\right) k = \frac{k^2}{2} \cdot$$

Hence $F(z) = 1 + \sum_{1}^{\infty} \frac{b_\nu c_\nu}{2} z^{\nu} \in P_K$ where $K = k^2/2$.

Theorem 4 — Let $f_1(z)$ and $f_2(z)$ belong to U_k and define $F(z)$ in E by the relation $F(z) = \lambda f_1(z) + (1 - \lambda) f_2(z)$, λ being any complex number such that $0 \leqslant \alpha = \arg \frac{\lambda}{1 - \lambda} < \pi$. Then $\text{Re} \left\{ z \frac{F'(z)}{F(z)} \right\} > 0$ if $r < R_c$ where R_c is the least positive root of the equation $1 - kr \sec \left(\frac{\alpha}{2} + k \sin^{-1} r \right) + r^2 = 0$.

PROOF : Let $F(z) = \lambda f_1(z) + (1-\lambda) f_2(z)$ where $f_1(z)$ and $f_2(z) \in U_k$.

Then

$$z\frac{F'(z)}{F(z)} = \frac{z\lambda f'_1(z) + z(1-\lambda) f'_2(z)}{\lambda f_1(z) + (1-\lambda) f_2(z)}$$

$$= z\frac{f'_1(z)}{f_1(z)}\left\{1 + \left\{\frac{\lambda}{1-\lambda}\frac{f_1(z)}{f_2(z)}\right\}^{-1}\right\}^{-1} + z\frac{f'_2(z)}{f_2(z)}\left\{1 + \frac{\lambda}{1-\lambda}\frac{f_1(z)}{f_2(z)}\right\}^{-1}.$$

Since $f_i(z) \in U_k$, $z\dfrac{f'_i(z)}{f_i(z)} \in P_k$ for $i = 1, 2$.

Hence $$\left|z\frac{f'_i(z)}{f_i(z)} - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{kr}{1-r^2}$$

and $\left|\arg\dfrac{f_i(z)}{z}\right| \leqslant k\sin^{-1} r$ for $i = 1, 2$; $|z| = r$ (Pinchuk 1969).

Applying the lemma (Stump 1971)

$$\operatorname{Re}\left\{z\frac{F'(z)}{F(z)}\right\} \geqslant \frac{1-r^2}{1+r^2} - \frac{kr}{1-r^2}\sec\beta/2$$

where

$$\beta = \arg\left\{\frac{\lambda}{1-\lambda}\frac{f_1(z)}{f_2(z)}\right\} = \arg\frac{\lambda}{1-\lambda} + \arg\frac{f_1(z)}{z} - \arg\frac{f_2(z)}{z}.$$

Then

$$|\beta| < \alpha + 2k\sin^{-1} r \text{ or } |\beta| \in [0, \pi) \text{ if } r < \sin\left(\frac{\pi-\alpha}{2k}\right).$$

If

$$r < \sin\left(\frac{\pi-\alpha}{2k}\right) \text{ then } 0 \leqslant |\beta| < \pi \text{ and } \sec\beta/2 \leqslant \sec\left(\frac{\alpha}{2} + k\sin^{-1} r\right).$$

Hence $\operatorname{Re}\left\{z\dfrac{F'(z)}{F(z)}\right\} > 0$ if $r < \min.\left[\sin\left(\dfrac{\pi-\alpha}{2k}\right), R_c\right]$

where R_c is the least positive value of r satisfying the equation

$$\frac{1+r^2}{1-r^2} - \frac{kr}{1-r^2}\sec\left(\frac{\alpha}{2} + k\sin^{-1} r\right) = 0$$

or $$1 - kr\sec\left(\frac{\alpha}{2} + k\sin^{-1} r\right) + r^2 = 0$$

which was to be shown. When $k = 2$ we get Stump's result (1971).

Theorem 5 — Let $f(z) \in U_k$. Then $F(z) = \{-f(z) \,.\, f(-z)\}^{+1/2}$ is also in U_k.

PROOF : Since $f(z) \in U_k$,

$$f(z) = z \exp \left\{- \int_0^{2\pi} \log (1 - ze^{-it})\, dm(t), \text{ where } m(t) \in M_k.\right.$$

Hence $F(z) = z \exp \left\{- \frac{1}{2} \int_0^{2\pi} \log (1 - ze^{-it})\, dm(t)\right\}$

$$\times \exp \left\{- \frac{1}{2} \int_0^{2\pi} \log (1 + ze^{-it})\, dm(t)\right\}$$

i. e. $z \dfrac{F'(z)}{F(z)} = 1 + \frac{1}{2} \displaystyle\int_0^{2\pi} \frac{ze^{-it}}{1 - ze^{-it}}\, dm(t) - \frac{1}{2} \int_0^{2\pi} \frac{ze^{-it}}{1 + ze^{-it}}\, dm(t)$

$$z \frac{F'(z)}{F(z)} = \frac{1}{2} \left\{ \frac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, dm(t) + \frac{1}{2} \int_0^{2\pi} \frac{1 - ze^{-it}}{1 + ze^{-it}}\, dm(t) \right\}$$

$$= \frac{1}{2} \{p(z) + p(-z)\} \text{ where } p(z) \in P_k.$$

Hence $z \dfrac{F'(z)}{F(z)} = \frac{1}{2} \left\{ \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z) \right\}$

$$+ \frac{1}{2} \left\{ \left(\frac{k}{4} + \frac{1}{2} \right) p_1(-z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(-z) \right\}$$

$$= \left(\frac{k}{4} + \frac{1}{2} \right) \left\{ \frac{p_1(z) + p_1(-z)}{2} \right\} - \left(\frac{k}{4} - \frac{1}{2} \right) \left\{ \frac{p_2(z) + p_2(-z)}{2} \right\}$$

where $p_i(z) \in P_2 (i = 1, 2)$. Also $\dfrac{p_i(z) + p_i(-z)}{2} \in P_2$ whenever $p_i(z) \in P_2$, $i = 1, 2$.

Hence

$$z \frac{F'(z)}{F(z)} = \left(\frac{k}{4} + \frac{1}{2} \right) q_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) q_2(z)$$

where $q_i(z) \in P_2$ for $i = 1, 2$. Hence

$$z \frac{F'(z)}{F(z)} \in P_k$$

which means $F(z) \in U_k$.

Theorem 6 — If $f(z) \in B_k$ then so does $F(z) = \frac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt.$

PROOF: Since $f(z) \in B_k$ we have $f'(z) \in P_k$.

Hence $f'(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)$ where $p_i(z) \in P_2$ for $i = 1, 2$. Integrating from 0 to z we get

$$f(z) = \left(\frac{k}{4} + \frac{1}{2}\right) f_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) f_2(z)$$

where $f_i(z) = \int_0^z \frac{p_i(t)}{t}\, dt,\ i = 1, 2.$

Also $F'(z) = \frac{f(z) - f(-z)}{2z}$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left\{\frac{f_1(z) - f_1(-z)}{2z}\right\} - \left(\frac{k}{4} - \frac{1}{2}\right)\left\{\frac{f_2(z) - f_2(-z)}{2z}\right\}.$$

Hence $F(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \int_0^z \frac{f_1(t) - f_1(-t)}{2t}\, dt - \left(\frac{k}{4} - \frac{1}{2}\right) \int_0^z \frac{f_2(t) - f_2(-t)}{2t}\, dt$

$$= \left(\frac{k}{4} + \frac{1}{2}\right) F_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) F_2(z).$$

Both $F'_1(z)$ and $F'_2(z)$ belong to P_2 (Singh 1970). Hence

$$F'(z) = \left(\frac{k}{4} + \frac{1}{2}\right) F'_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) F'_2(z)$$

belongs to P_k.

Thus $F(z) \in B_k$.

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