ROTATION OF EIGENVECTORS BY A PERTURBATION

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(Communicated by F. C. Auluck, F.N.A.)

(*Received* 29 *April* 1974)

The purpose of this note is to estimate the gap that separates the eigensubspace of the unperturbed operator and that of the perturbed operator acting on n-dimensional complex Hilbert space.

§1. When a Hermitian linear operator acting on n-dimensional complex Hilbert space $\mathscr{H}$ is slightly perturbed, by how much can its invariant subspace change? Given an estimate for the gap that separates the cluster of neighbouring eigenvalues from all other eigenvalues, how much can the subspace spanned by the eigenvectors differ from the subspace spanned by our approximations? These questions are closely related. Davis (1963) has answered these questions employing the techniques of soft analysis when the perturbed operator is also Hermitian. The purpose of this note is to give a function theoratic treatment of the problem when the perturbed operator is normal. We shall establish the terminology below.

§2. $\mathscr{H}$ is a finite dimensional inner product space over complex numbers with $(.\,,.)$ denoting the inner product and that A is a self-adjoint operator in $\mathscr{H}$ i.e. $A = A^*$. We shall assume that A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $\lambda_{i+1} \leqslant \lambda_i$ counting multiplicity and orthogonal eigenvectors x_i. Let $\mu_1, \mu_2, \ldots, \mu_k$ with $\mu_{i+1} \leqslant \mu_i$ be the eigenvalues not counting multiplicity. Then A can be written as

$$A = \sum_{i=1}^{k} \mu_i P_i \quad \text{where} \quad \sum_{i=1}^{k} P_i = I.$$

H is a transformation in $\mathscr{H}$ such that $A + H = N$ is normal. The eigenvalues of $A + H = N$ will be denoted by $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$ counting multiplicity and associated orthogonal vectors by x'_i. Let $\mu'_1, \mu'_2, \ldots, \mu'_j$ be the eigenvalues of N not counting multiplicity. Then N can be written as

$$N = \sum_{i=1}^{j} \mu_i P'_i \quad \text{where} \quad \sum_{i=1}^{j} P'_i = I.$$

we choose the eigenvector x_i' of the perturbed operator corresponding to the eigenvector x_i of the unperturbed operator in the form

$$x_i' = \frac{P' x_i}{\| P' x_i \|}$$

where P' is the projection on 1-dimensional subspace of eigenvectors corresponding to x_i'. We set

$$\theta_i = \arccos (x_i', x_i).$$

More generally, let the spectrum of A be confined to m intervals of length $\leqslant 2\beta$ with gaps between the lengths $\geqslant \gamma > 0$. Let P_j denote the spectral projection corresponding to $[\nu_j, \mu_j]$ $(j = 1, \ldots, m)$,

$$0 \leqslant \mu_j - \nu_j \leqslant 2\beta, \quad \gamma \leqslant \nu_{j+1} - \mu_j, \quad \sum_{j=1}^{m} P_j = I,$$

and let $\| H \| = \delta < \gamma/2$. Then it is well known that each P_j' denotes the spectral projection corresponding to the oval shaped region. It is well known that the dimensionality of P_j' is the same as that of P_j for each $j = 1, 2, \ldots, m$. Define $U = U(\{P_j\}, \{P_j'\})$ by requiring for all $j = 1, 2, \ldots, m$ that

$$UP_j = \left(P_j' \, P_j \, P_j'\right)^{-1/2} P_j' \, P_j$$

(Riesz and Nagy 1955, p. 136; Davis 1958; Kato 1966)

$$\text{i.e. } U = \sum_{j=1}^{m} \oplus \left(P_j' \, P_j \, P_j'\right)^{-1/2} P_j' \, P_j.$$

This makes sense and determines a unitary U at least under the hypothesis that for all j, $x = P_j x \neq 0 \Rightarrow P_j' x \neq 0$. This hypothesis is satisfied if the P_j and P_j' arise from A and N as described above. Clearly

$$UP_j = P_j' U = P_j' (P_j' \, P_j \, P_j')^{-1/2} \, P_j.$$

Davis (1958) has proved that such a rotation U is most economical.

The singularities of $R(\zeta, A) = (A - \zeta I)^{-1}$ are exactly the eigenvalues $\lambda_1, \ldots, \lambda_k$ of A. Let us consider the Laurent series of $R(\zeta, A)$ at $\zeta = \lambda_h$.

$$R(\zeta, A) = \sum_{-\infty}^{\infty} (\zeta - \lambda_h)^n A_n.$$

The coefficients A_n are given by

$$A_n = \frac{1}{2\pi i} \int_\Gamma (\zeta - \lambda_h)^{-n-1} R(\zeta, A)\, d\zeta$$

where Γ is a positively oriented small circle enclosing $\zeta = \lambda_h$ but excluding other eigenvalues of A. Since Γ may be expanded to slightly larger circle Γ' without changing A_n, we have

$$A_n A_m = \frac{1}{(2\pi i)^2} \int_{\Gamma'} \int_\Gamma (\zeta - \lambda_h)^{-n-1} (\zeta' - \lambda_h)^{-m-1} R(\zeta, A) R(\zeta', A)\, d\zeta\, d\zeta'.$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_\Gamma (\zeta - \lambda_h)^{-n-1} (\zeta' - \lambda_h)^{-m-1} [R(\zeta') - R(\zeta)]\, (\zeta' - \zeta)\, d\zeta\, d\zeta'$$

using the resolvent equation. The double integral on the right may be computed in any order. Considering that Γ' lies outside Γ, we have

$$\frac{1}{2\pi i} \int_\Gamma (\zeta - \lambda_h)^{-n-1} (\zeta' - \zeta)^{-1}\, d\zeta = \eta_n (\zeta' - \lambda_h)^{-n-1}$$

$$\frac{1}{2\pi i} \int_{\Gamma'} (\zeta' - \lambda_h)^{-m-1} (\zeta' - \zeta)^{-1}\, d\zeta' = (1 - \eta_m)\, (\zeta - \lambda_h)^{-m-1}$$

where the symbol η_n is defined by

$$\eta_n = \begin{cases} 1 \text{ if } n \geqslant 0 \\ 0 \text{ if } n < 0. \end{cases}$$

Thus $$A_n A_m = \frac{\eta_n + \eta_m - 1}{2\pi i} \int_\Gamma (\zeta - \lambda_h)^{-n-m-2} R(\zeta, A)\, d\zeta$$

$$= (\eta_n + \eta_m - 1)\, A_{n+m+1}.$$

For $n = m = -1$, this gives $A_{-1}^2 = -A_{-1}$. Thus $-A_{-1}$ is a projection which we shall denote by P_h.

For negative values of n, m, above equation gives

$$A_{-2}^2 = -A_{-3},\ A_{-2}\, A_{-3} = -A_{-4}.$$

On setting $-A_{-2} = D_h$, we thus obtain $A_{-k} = -D_h^{k-1}$, $k \geqslant 2$. Similarly we obtain $A_n = S_h^{n+1}$ for $n \geqslant 0$ with $S_h = A_0$. Thus the Laurent series takes the form

$$R(\zeta, A) = -(\zeta - \lambda_h)^{-1} P_h - \sum_{n=1}^{\infty} (\zeta - \lambda_h)^{-n-1} D_h^n$$

$$+ \sum_{n=0}^{\infty} (\zeta - \lambda_h)^n S_h^{n+1}.$$

The following identities are well known :

(i) $P_h D_h = D_h P_h = D_h, \; P_h S_h = S_h P_h = 0$

(ii) $P_h P_k = \delta_{hk} P_h, \; \sum_{h=1}^{m} P_h = I, \; P_h A = A P_h.$

$$P_h = -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta) \, d\zeta$$

where Γ_h denotes the circle which includes the eigenvalue λ_h and no other.

(iii) For any simple closed (rectifiable) curve Γ with positive direction and not passing through any eigenvalue λ_h', we have

$$\frac{1}{2\pi i} \int_{\Gamma} R(\zeta) \, d\zeta = -\Sigma P_h$$

where the sum is taken for those h for which λ_h is inside Γ.

(iv) $(A - \lambda_h) S_h = S_h (A - \lambda_h) = I - P_h$

$P_h(A - \lambda_h) = (A - \lambda_h) P_h = D_h.$

(v) If A is normal, then $\| R(\zeta, A) \| \leqslant \dfrac{1}{\text{dist}\,(\zeta, \text{Spectrum } A)}$

(vi) If A is normal, then $\| S_h \| = \dfrac{1}{\min_{h \neq k} | \lambda_h - \lambda_k |}$.

§3. *A Bound for the rotation of a single spectral subspace* — $P = P_j$ denotes the spectral projection corresponding to the eigenvalues of A lying in the interval $[\nu_j, \mu_j]$ where $\mu_j - \nu_j \leqslant 2\beta$. The intervals $(\nu_j - \gamma, \nu_j)$ and $(\mu_j, \mu_j + \gamma)$ contain no eigenvalues of A. Without essential change in the argument, we shall take $-\nu_j = \mu_j = \beta$. The following theorem gives an estimate for $\| P' - P \|$ where $P' = P_j'$ is the corresponding spectral projection of $N = A + H$.

Theorem 1 — If $\| H \| \leqslant \delta < \gamma/2$, then

$$\| P - P' \| \leqslant \frac{\delta(2\beta+\gamma)}{\pi\gamma} \left\{ \frac{4Q_0}{\gamma - 2\delta} + \frac{2(\pi - 2Q_0)}{(4\beta\gamma+\gamma^2)^{1/2} - 2\delta} \right\} \leqslant \frac{2\delta(2\beta+\gamma)}{\gamma(\gamma-2\delta)}$$

where

$$Q_0 = \arc\cos \frac{2\beta}{2\beta+\gamma}.$$

Proof : The opening remark is trivial.

$$(A - \zeta I)^{-1} - (A + H - \zeta I)^{-1} = (A - \zeta I)^{-1} H (A + H - \zeta I)^{-1}.$$

Now

$$P = -\frac{1}{2\pi i} \int_\Gamma (A - \zeta I)^{-1} \, d\zeta$$

$$P' = -\frac{1}{2\pi i} \int_\Gamma (A + H - \zeta I)^{-1} \, d\zeta$$

where Γ is a circle with centre origin and radius $\beta + \delta + K$, $K = \gamma - 2\delta$.

$$P - P' = -\frac{1}{2\pi i} \int_\Gamma \{(A - \zeta I)^{-1} - (A + H - \zeta I)^{-1}\} \, d\zeta$$

$$= -\frac{1}{2\pi i} \int_\Gamma (A - \zeta I)^{-1} H (A + H - \zeta I)^{-1} \, d\zeta$$

$$\| P - P' \| \leqslant \frac{1}{2\pi} \| H \| \, \| (A - \zeta I)^{-1} \| \int_\Gamma \| (A + H - \zeta I)^{-1} \| \, | d\zeta |$$

and the integral on the right equals

$$2(\beta + \delta + K) \left\{ \frac{2}{K} Q_0 + \frac{\pi - 2Q_0}{[(\beta+\delta+K)^2 - \beta^2]^{1/2} - \delta} \right\}$$

where

$$Q_0 = \arc\cos \frac{2\beta}{2\beta+\gamma}.$$

Substituting for K, we obtain

$$\| P - P' \| \leqslant \frac{\delta(2\beta+\gamma)}{\pi\gamma} \left\{ \frac{4Q_0}{\gamma - 2\delta} + \frac{2(\pi - 2Q_0)}{(4\beta\gamma+\gamma^2)^{1/2} - 2\delta} \right\}$$

$$\leqslant \frac{\delta(2\beta+\gamma)}{\pi\gamma} \left\{ \frac{4Q_0}{\gamma - 2\delta} + \frac{2\pi - 4Q_0}{\gamma - 2\delta} \right\} = \frac{2\delta(2\beta+\gamma)}{\gamma(\gamma-2\delta)}.$$

Suppose $\beta = 0$, i.e. the reducing subspace P_h in view corresponds to a single eigenvalue (which is taken to be zero), but need not be one dimensional. In this case, we have the following estimate.

Theorem 2 — $\| P - P' \| \leqslant \dfrac{2\delta}{\gamma - \delta}$.

PROOF: $$(A + H) P' = - \frac{1}{2\pi i} \int_{\Gamma} (A + H)(A + H - \zeta I)^{-1} \, d\zeta$$

where Γ is a circle about O of radius δ.

Since $$(A + H)(A + H - \zeta I)^{-1} = (A + H - \zeta I + \zeta I)(A + H - \zeta I)^{-1} = I + \zeta (A + H - \zeta I)^{-1}$$

and since the integral of I along Γ vanishes, therefore,

$$(A + H) P' = - \frac{1}{2\pi i} \int_{\Gamma} \zeta (A + H - \zeta I)^{-1} \, d\zeta.$$

Also $(A + H) P' = A(P' - P) + HP'$, hence

$$S(A + H) P' = (I - P)(P' - P) + SHP' = (I - P)(P' - P) + SH(P' - P) + SHP.$$

Thus $- SHP + S(A + H) P' = (I + SH)(P' - P)$, since $P(P' - P) = 0$.

$$(I + SH)(P' - P) = - SHP - \frac{S}{2\pi i} \int_{\Gamma} (A + H - \zeta I)^{-1} \, d\zeta$$

$$\| P' - P \| \leqslant \| (I + SH)^{-1} \| \left\{ \| SH \| + \frac{\| S \|}{2\pi} \cdot 2\pi\delta \right\}$$

$$\leqslant (1 - \| S \| \delta)^{-1} 2\delta \| S \| = \left(1 - \frac{\delta}{\gamma}\right)^{-1} \frac{2\delta}{\gamma} = \frac{2\delta}{\gamma - \delta}.$$

Suppose $\beta = 0$, i.e., the reducing subspace P_h in view corresponds to a single eigenvalue which is taken to be zero. Furthermore, this eigenvalue is 1-dimensional. In this case, we have the following estimate.

Theorem 3 — $\| P' - P \| \leqslant \dfrac{\delta}{\gamma - 2\delta}$.

PROOF: Let λ' and x' denote the corresponding value and the eigenvector of N. Recall that the $x' = \dfrac{P'x}{\| P'x \|}$.

$$(A + H - \lambda') P' = 0$$

i.e. $A(P' - P) + (H - \lambda') P' = 0$, using $AP = 0$.

Multiplying on the right by S, we obtain

$$(I - P)(P' - P) + S(H - \lambda')(P' - P + P) = 0$$

i.e. $$(P' - P) + S(H - \lambda')(P' - P) = - SHP.$$

Thus $[I + S(H - \lambda')](P' - P) = - SHP$

or $P' - P = - [I + S(H - \lambda')]^{-1} SHP$.

Hence $\| P' - P \| \leqslant \dfrac{\| S \| \delta}{1 - 2\delta \| S \|} = \dfrac{\delta}{\gamma - 2\delta}$.

§4. *The total amount of rotation* — Using the estimates obtained in Theorem 2 and Theorem 3, the following estimates result.

Theorem 4 — Assume A has eigenvalues any two of which differ by at least γ, and assume $\| H \| = \delta < \gamma/2$. Then

$$\| I - U \| \leqslant \frac{2m\delta}{\gamma - \delta}.$$

PROOF : $$\| I - U \| = \left\| \sum_{i=1}^{m} (P_i - UP_i) \right\|$$

$$\leqslant \sum_{i=1}^{m} \| P_i - UP_i \| = \sum_{i=1}^{m} \left\| P_i - P_i'(P_i'.P_i.P_i')^{-1/2} P_i' \right\|$$

$$\leqslant \sum_{i=1}^{m} \left\| P_i - P_i' \right\| = \frac{2\delta}{\gamma - \delta} m.$$

Theorem 5 — Let A have simple eigenvalues, any two differing by at least γ and let $\| H \| = \delta < \gamma/2$, then

$$\| I - U \| \leqslant \frac{n\delta}{\gamma - 2\delta}.$$

PROOF : Proceeding as in the above proof, we obtain

$$\| I - U \| \leqslant \sum_{i=1}^{n} \| P_i - P_i' \| = \frac{n\delta}{\gamma - 2\delta}.$$

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