ON COMMON FIXED POINTS III

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A few sufficient conditions for the existence of a unique common fixed point of a finite number of self mappings of a topological space have been obtained.

Let X be a metric space. A mapping $T : X \to X$ is called a contraction mapping if $d(Tx, Ty) \leqslant kd(x, y)$ for all x, y in X where k is a real constant such that $0 \leqslant k < 1$. The contraction principle of Banach guarantees a unique fixed point of each contraction mapping of a complete metric space into itself. The aim of this note is to prove some fixed point theorems in an arbitrary topological space, which is not necessarily a metric space.

Theorem 1 — Let X be a Hausdorff space and $T_1, T_2, \ldots, T_r$ be mappings of X into itself such that $T_iT_j = T_jT_i \ \forall \ x \in X$ and $T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r}$ is also continuous self map of X where $p_1, p_2, \ldots, p_r$ are positive integers. Let $F : X \times X \to [0, \infty)$ be continuous such that

(A) $F\left(T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} x, T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} y\right) \leqslant F(x, y)$ for all x, y in X and when $x \neq y$ there is some n depending on x and y such that

(B) $F\left(\left(T_1^{p_1} \ldots T_r^{p_r}\right)^n x, \left(T_1^{p_1} \ldots T_r^{p_r}\right)^n y\right) < F(x, y)$

(C) there exists a $x \in X$ such that $\left\{\left(T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r}\right)^n x\right\}$ has a convergent subsequence, then $T_1, T_2, \ldots, T_r$ have a unique common fixed point.

PROOF: Set $T = T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r}$ and define $x_1 = Tx$ and for $n > 1$, $x_n = Tx_{n-1}$.

Then from (A) we get a monotone a sequence of non-negative real numbers

$$F(x, x_1) \geqslant F(x_1, x_2) \geqslant \ldots \geqslant F(x_n, x_{n+1}) \geqslant \ldots$$

which must converge along with all its subsequences to some real number η, say. Also we have a convergent subsequence $\left\{T_x^{n_k}\right\}$, where $T = T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r}$, in X which converges to some z_0 in X. For some n depending on z_0, Tz_0.

$$F(T^n z_0, T^{n+1} z_0) < F(z_0, Tz_0) \text{ if } z_0 \neq Tz_0.$$

Again since F and T are continuous, we have

$$\begin{aligned} F(z_0, Tz_0) &= F(\lim_k T^{n_k}x, \lim T^{n_{k+1}}x) \\ &= \lim_k F(T^{n_k}x, T^{n_{k+1}}x) \\ &= \eta \\ &= \lim_k F(T^{n_k+n}x, T^{n_k+n+1}x) \\ &= F(T^n z_0, T^{n+1} z_0). \end{aligned}$$

So we arrive at a contradiction if $z_0 \neq Tz_0$. If y_0 ($y_0 \neq z_0$) be another fixed point of T then for some m depending on y_0, z_0 we have $F(z_0, y_0) > F(T^m z_0, T^m y_0)$. But $z_0 = Tz_0$ and $y_0 = Ty_0$. Hence $z_0 = Tz_0 = T^2 z_0 = \ldots = T^m z_0$ and similarly $y_0 = T^m y_0$ and so $F(z_0, y_0) > F(z_0, y_0)$, a contradiction. Hence z_0 is a unique fixed point of T. We shall show now that

$$T_1 z_0 = T_2 z_0 = \ldots = T_r z_0 = z_0.$$

Now we have $Tz_0 = T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} z_0 = z_0$

$$\Rightarrow T_1 \left(T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} \right) z_0 = T_1 z_0$$

or $\quad T_1 T z_0 = T_1 z_0.$

But $\quad T_i T_j = T_j T_i \ \forall \ x \in X.$

So $\quad T_1 T z_0 = T_1 z_0$

$$\Rightarrow T T_1 z_0 = T_1 z_0.$$

But z_0 is a unique fixed point of T. Hence $T_1 z_0 = z_0$ and proceeding in this way one can similarly establish that $T_2 z_0 = T_3 z_0 = \ldots = T_r z_0 = z_0$. This completes the proof.

Theorem 2 — Let X be a Hausdorff space and let the mappings $T_1, T_2, \ldots, T_r$ of X into itself and the mapping F as defined in Theorem 1 satisfy condition (B) and (C) of Theorem 1 and for all $x, y \in X$ let F satisfy:

$$(\mathrm{A}^*)\ F\left(T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} x,\ T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} y\right)$$
$$\leqslant \alpha F\left(x,\ T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} x\right) + \beta F\left(y,\ T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} y\right)$$
$$+ \gamma F(x, y),\ p_1, p_2, p_3, \ldots, p_r \text{ are positive integers,}$$
$$\alpha > 0,\ \beta > 0,\ \gamma > 0,\ \alpha + \beta + \gamma < 1.$$

Then there exists a unique common fixed point of $T_1, T_2, \ldots, T_r$.

PROOF : we define $\{x_n\}$ as in Theorem 1. Then from (A*) we have

$$F(x, x_1) > F(x_1, x_2) > \ldots > F(x_n, x_n + 1)$$

which must converge along with all its subsequences to some real number λ. The rest of the proof is as demonstrated in Theorem 1.

Corollary — If X is a metric space with metric d and $T_1, T_2, T_3, \ldots, T_r$ are mappings of X into itself such that

$$T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r} : X \to X$$

is a non-expansive mapping such that for each $x, y \in X$ with $x \neq y$ there is an n such that

$$d\left(\left(T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r}\right)^n x, \left(T_1^{p_1} T_2^{p_2} \ldots T_r^{p_r}\right)^n y\right) < d(x, y).$$

If for some $x \in X$ the sequence $\{x_n\}$ as defined above has a convergent subsequence $\{x_{nk}\}$ converging to $z_0 \in X$, then z_0 is the unique common fixed point of $T_1, T_2, \ldots, T_r$.

PROOF : All the assumptions of Theorem 1 are satisfied with d playing the role of F.

Definition 1 — If X is a metric space and $A \subseteq X$ is a bounded subset of X, let $\alpha(A)$ be the inifimum of all $\epsilon > 0$ such that a finite number of open spheres of diameter $< \epsilon$ will conver A (Kuratowski 1952).

Definition 2 — If $T : X \to X$ is such that for each bounded subset $A \subseteq X$ with $\alpha(A) \neq 0$ we get $\alpha(TA) < \alpha(A)$, then T is called densifying (Furi and Vignoli 1969). It is easily seen that (Singh 1971)

(1) $0 \leqslant \alpha(A) \leqslant D(A)$, the diameter of A,

(2) $\alpha(A) = 0$ and X complete imply A is compact,

(3) If A is compact then $\alpha(A) = 0$,

(4) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,

(5) If $\bar{A}$ is the closure of A then
$\alpha(\bar{A}) = 0 \Leftrightarrow \alpha(A) = 0$ (Szufla 1968).

Theorem 4 — Let (X, d) be a complete metric space and let $F : X \times X \to [0, \infty)$ be continuous. Let T_1 and T_2 be two selfmappings of X such that $T_1^p T_2^q$ is a continuous mapping of X into itself and $T_1^p T_2^q$ is densifying and for all $x, y \in X$,

$$F\left(T_1^p T_2^q x, T_1^p T_2^q y\right) \leqslant F(x, y)$$

and in case $x \neq y$ there is some n such that $F(x_n, y_n) < F(x, y)$

where $\quad x_n = \left(T_1^p T_2^q\right)^n x, \; y_n = \left(T_1^p T_2^q\right)^n y.$

If for some $x \in X$ the sequence $\{x_n\}$

where $\quad x_1 = T_1^p T_2^q x$ and for $n > 1$, $x_n = \left(T_1^p T_2^q\right) x_{n-1}$ is bounded, then T_1 and T_2 have a unique common fixed point if $T_1 T_2 = T_2 T_1$ for all x, y in X.

PROOF : Let $T = T_1^p T_2^q$ and $A = \bigcup_{n=0}^{\infty} \{x_n\}$ and let $\bar{A}$ denote the closure of A (Singh 1971). We will show that $\bar{A}$ is compact which, by completeness of X, will be true if $\alpha(\bar{A}) = 0$. Let $\alpha(\bar{A}) > 0$ or equivalently suppose that $\alpha(A) > 0$.

Since T is densifying we have $\alpha(T(A)) < \alpha(A)$.

But
$$\begin{aligned} \alpha(A) &= \max.\{\alpha(T(A)), \alpha(x_0)\} \\ &= \max.\{\alpha(TA), 0\} \\ &= \alpha(T(A)). \end{aligned}$$

This contradiction gives $\alpha(\bar{A}) = 0$ and so $\bar{A}$ is compact. Now since T is continuous we have $T(\bar{A}) \subseteq \overline{T(A)} \subseteq \bar{A}$. So the space $\bar{A}$ with $T : \bar{A} \to \bar{A}$ now satisfies all the assumptions of Theorem 1 and therefore there exists a fixed point $z_0 \in \bar{A}$. From the condition $F(x, y) > F(T^n x, T^n y)$ for some n it follows that z_0 is a unique fixed point of T. The rest of the proof follows from Theorem 2.

The above results generalize a theorem of Singh (1971).

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