ON THE SATURATION OF CLASSES OF FUNCTIONS BY (N, p_n, q_n) METHOD

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Zamanski (1949) determined a saturation class by considering $(C, 1)$ means of the Fourier series $f(x)$. We have determined the class of saturation for generalized Nörlund means of the Fourier series $f(x)$ with order $\left(\frac{r_n}{R_n}\right)$.

§1. Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let its Fourier series be given by

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x). \qquad \ldots(1.1)$$

Then

$$\sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx) = \sum_{k=1}^{\infty} B_k(x) \qquad \ldots(1.2)$$

is called the conjugate series of f. Let $\tilde{S}_n(f, x)$, $n = 1, 2, \ldots$ denote the partial sum of (1.2) associated with $f(x)$. We have

$$\tilde{D}_n(x) = \sin x + \sin 2x + \ldots + \sin nx$$

$$= \frac{\cos x/2 - \cos\left(\frac{n+1}{2}\right)x}{\sin\frac{1}{2}x}. \qquad \ldots(1.3)$$

Since $\quad B_k(x) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x+t) \sin kt \, dt,$

we obtain

$$\tilde{S}_n(f, x) = \frac{1}{\pi}\int_0^{\pi} [f(x+t) - f(x-t)]\, \tilde{D}_n(t)\, dt. \qquad \ldots(1.4)$$

We shall say,

$$\tilde{f}(x) = \frac{1}{2\pi} \int_0^{\pi} [f(x+t) - f(x-t)] \cot \tfrac{1}{2}t \, dt \qquad \ldots(1.5)$$

is the conjugate function of f, if the integral (1.5) converges absolutely for all $f(x) \in K$ (where K is a class of functions) and if

$$\int_0^{\pi} | f(x+t) - f(x-t) | \cot \frac{t}{2} \, dt$$

is an integrable function.

§2. Let $\{p_n\}$, $\{q_n\}$ be non-negative, non-increasing generating sequences for (N, p_n, q_n) method such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty \text{ as } n \to \infty. \qquad \ldots(2.1)$$

$$Q_n = q_0 + q_1 + \ldots + q_n \qquad \ldots(2.2)$$

and

$$R_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 \to \infty \text{ as } n \to \infty \qquad \ldots(2.3)$$

A given series $\sum_{n=0}^{\infty} g_n$ with the sequence of partial sums $\{S_n\}$ is said to be summable (N, p_n, q_n) to g provided that

$$t_n^{p\cdot q}(f, x) = \frac{1}{R_n} \sum_{k=0}^{n} R_{n-k}\, g_k$$

$$= \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k}\, S_k \to g \text{ as } n \to \infty \qquad \ldots(2.4)$$

and $t_n^{p\cdot q}(f, x)$ is called the Nörlund operator.

We define the norm

$$\| f(x) - t_n^{p\cdot q}(f, x) \| = \max_{0 \leqslant x \leqslant 2\pi} | f(x) - t_n^{p\cdot q}(f, x) | .$$

If there exists a positive non-increasing function $\phi(n)$ and a class of functions K, with the following properties

(i) $\| f(x) - t_n^{p \cdot q}(f, x) \| = o(\phi(n)) \Rightarrow f(x)$ is constant,

(ii) $\| f(x) - t_n^{p \cdot q}(f, x) \| = O(\phi(n)) \Rightarrow f(x) \in K$,

and

(iii) $f(x) \in K \Rightarrow \| f(x) - t_n^{p \cdot q}(f, x) \| = O(\phi(n))$.

Then the Nörlund operators are saturated with the order $\phi(n)$ and the class K.

In this paper we prove that the above method of summation is saturated with the order r_n/R_n and that the class K consists of all continuous functions f such that $\tilde{f} \in$ Lip 1 where $\tilde{f}$ is conjugate function of f.

§3. We shall prove the following theorem :

Theorem — Let $\{r_n\}$ be a sequence of positive constants satisfying the following conditions :

$$\frac{r_{n-k}}{r_n} \to 1 \text{ as } n \to \infty \text{ for fixed } k \leqslant n \qquad \ldots(3.1)$$

$$R_n = O(nr_n) \qquad \ldots(3.2)$$

$$\sum_{k=0}^{n} \frac{| r_{n-k} - r_{n-k-1} |}{k+1} = O\left(\frac{r_n}{n}\right) \qquad \ldots(3.3)$$

then the saturation class of operators $t_n^{p \cdot q}(f, x)$ relative to r_n, consists of all continuous functions f for which $\tilde{f} \in$ Lip 1 and the order of saturation is $\frac{r_n}{R_n}$.

Proof of the Theorem : Firstly we show that if $\| f(x) - t_n^{p \cdot q}(f, x) \| = o\left(\frac{r_n}{R_n}\right)$ and (3.1) is satisfied then f is constant.

Using (2.4) we get

$$\frac{1}{\pi} \int_{-\pi}^{\pi} t_n^{p \cdot q}(f, x) \cos qx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{k=0}^{n} \frac{R_{n-k}}{R_n} A_k(x) \cos qx \, dx$$

$$= \frac{1}{\pi} \sum_{k=0}^{n} \frac{R_{n-k}}{R_n} \int_{-\pi}^{\pi} A_k(x) \cos qx \, dx$$

(*equation continued on p.* 1264)

$$= \frac{1}{\pi} \sum_{k=0}^{n} \frac{R_{n-k}}{R_n} \int_{-\pi}^{\pi} a_q \cos kx \cos qx \, dx$$

$$= \frac{1}{\pi} \frac{R_{n-q}}{R_n} \int_{-\pi}^{\pi} a_q \cos^2 qx \, dx$$

$$= \frac{1}{\pi} \frac{R_{n-q}}{R_n} . \pi a_q = \frac{R_{n-q}}{R_n} a_q .$$

Thus

$$a_q - \frac{R_{n-q}}{R_n} . a_q = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos qx \, dx - \frac{1}{\pi} \int_{-\pi}^{\pi} t_n^{p \cdot q} (f, x) \cos qx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos qx \, [f(x) - t_n^{p \cdot q} (f, x)] \, dx.$$

Hence

$$\left| a_q - \frac{R_{n-q}}{R_n} . a_q \right| < \| f(x) - t_n^{p \cdot q} (f, x) \| . \frac{1}{\pi} \int_{-\pi}^{\pi} 1 . \, dx$$

$$= o\left(\frac{r_n}{R_n}\right)$$

$$a_q \left| 1 - \frac{R_{n-q}}{R_n} \right| = a_q \sum_{p=0}^{q-1} \frac{r_{n-p}}{R_n} = o\left(\frac{r_n}{R_n}\right)$$

$$= a_q \frac{\sum_{p=0}^{q-1} r_{n-p}}{r_n} = o(1),$$

taking the limits of both the sides as $n \to \infty$ and using (3.1) we get $q \cdot a_q = 0$ which implies that $a_q = 0$ for each $q - 1 \geqslant 0$, i.e., $q \geqslant 1$. Similarly we can show that $b_q = 0$ for each $q - 1 \geqslant 0$, i.e., $q \geqslant 1$. Hence $f(x) = \frac{1}{2} a_0$, a constant.

Secondly if $\| f(x) - t_n^{p \cdot q} (f, x) \| = O\left(\frac{r_n}{R_n}\right)$ and (3.1) is satisfied, then we will show that $\tilde{f}(x) \in \text{Lip } 1$.

Suppose $T_n(x) = f(x) - t_n^{p\cdot q}(f, x)$. Then $\| T_n(x) \| = O\left(\frac{r_n}{R_n}\right)$. Now taking the Nth arithmetic mean $\sigma_N[x; T_n]$ of the series

$$T_n(x) \sim \sum_{k=1}^{\infty} \left(1 - \frac{R_{n-k}}{R_n}\right) A_k(x)$$

we have

$$\sigma_N(x; T_n) = \sum_{k=1}^{N} \left(1 - \frac{R_{n-k}}{R_n}\right)\left(1 - \frac{k}{N+1}\right) A_k(x). \qquad \ldots(3.4)$$

Now it is known that $\| T_n \| > \| \sigma_N[x; T_n] \|$ and so by hypothesis

$$\left\| \sum_{k=1}^{N} \left(1 - \frac{R_{n-k}}{R_n}\right)\left(1 - \frac{k}{N+1}\right) A_k(x) \right\| = O\left(\frac{r_n}{R_n}\right) \text{ for } N \leqslant n$$

or

$$\left\| \sum_{k=1}^{N} \frac{\left(1 - \frac{R_{n-k}}{R_n}\right)}{\left(\frac{r_n}{R_n}\right)} \left(1 - \frac{k}{N+1}\right) A_k(x) \right\| = O(1) \text{ for } N \leqslant n.$$

Taking the limit as $n \to \infty$ and using (3.1) we get

$$\left\| \sum_{k=1}^{N} k\, A_k(x) \left(1 - \frac{k}{N+1}\right) \right\| = O(1) \qquad \ldots(3.5)$$

The left-hand side of the above equation represents $(C, 1)$ means of the Fourier series $\sum_{k=1}^{\infty} - k\, A_k(x)$, since $- k\, A_k(x) = B_k'(x)$, therefore (3.5) is equivalent to

$$\| \tilde{\sigma}_N'(f) \| < M \qquad \ldots(3.6)$$

$\left(\tilde{\sigma}_N'(f)\right.$ represents the $(C, 1)$ means of the conjugate series$\left.\right)$. Applying mean value theorem, from (3.6) we get

$$| \tilde{\sigma}_N(f, x + t) - \tilde{\sigma}_N(f, x) | = O(|t|) \text{ as } t \to 0.$$

Since $$\begin{aligned} |\tilde{f}(x+t)-\tilde{f}(x)| &< |\tilde{f}(x+t)-\tilde{\sigma}_N(f, x+t)| \\ &\quad + |\tilde{\sigma}_N(f, x+t)-\tilde{\sigma}_N(f, x)| \\ &\quad + |\tilde{\sigma}_N(f, x)-\tilde{f}(x)| \\ &= o(1)+O(|t|)+o(1) \\ &= O(|t|). \end{aligned}$$

Therefore $\tilde{f} \in \text{Lip } 1$.

'Conversely' suppose (3.2) and (3.3) is satisfied and $\tilde{f} \in \text{Lip } 1$ then we will prove that

$$\| f(x) - t_n^{p\cdot q}(f, x) \| = O\left(\frac{r_n}{R_n}\right).$$

Now $$\tilde{S}_n(\tilde{f}, x) = \frac{1}{\pi}\int_0^{\pi} [f(x+t)-f(x-t)] \frac{\cos\frac{t}{2} - \cos\left(\frac{n+1}{2}\right)t}{2\sin(t/2)}\, dt$$

where $\tilde{S}_n(\tilde{f}, x)$ denotes the partial sum of the conjugate series associated with $\tilde{f}(x)$, we have

$$\begin{aligned} t_n^{p\cdot q}(\tilde{S}_n(\tilde{f}, x)) &= \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} \tilde{S}_k(\tilde{f}, x) \\ &= \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} \frac{1}{2\pi} \int_0^{\pi} [f(x+t)-f(x-t)] \cot(t/2)\, dt \\ &\quad - \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} \frac{1}{2\pi} \int_0^{\pi} [f(x+t)-\tilde{f}(x-t)] \frac{\cos(k+\frac{1}{2})t}{\sin(t/2)}\, dt. \end{aligned}$$

Since $\tilde{f} \in \text{Lip } 1$, $-f+\frac{1}{2}a_0$ is identical to $\tilde{\tilde{f}}$, therefore

$$\begin{aligned} | f(x) - t_n^{p\cdot q}(f, x) | = \Bigg| \frac{1}{2\pi} \int_0^{\pi} & [\tilde{f}(x+t)-\tilde{f}(x-t)] \\ & \times \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} \frac{\cos(k+\frac{1}{2})t}{\sin(t/2)}\, dt \Bigg| \end{aligned}$$

(*equation continued on p.* 1267)

$$\leqslant \frac{1}{2\pi} \int_0^{\pi/n} | \tilde{f}(x+t) - \tilde{f}(x-t) |$$

$$\times \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} \frac{| \cos (k+\frac{1}{2})t |}{\sin (t/2)} \, dt$$

$$+ \frac{1}{2\pi} \int_{\pi/n}^{\pi} | \tilde{f}(x+t) - \tilde{f}(x-t) | \frac{1}{\sin (t/2)}$$

$$\times \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} | \cos (k+\tfrac{1}{2})t | \, dt$$

$$= I_1 + I_2 \text{ (say)}.$$

Applying $| \tilde{f}(x+t) - \tilde{f}(x-t) | < K | t |$ (K is constant), we have

$$I_1 < \sum_{k=0}^{n} \frac{r_{n-k}}{R_n} \int_0^{\pi/n} \frac{2K | t |}{| t |} \, dt = O(1/n)$$

$$= O\left(\frac{r_n}{R_n}\right) \text{ from (3.2).}$$

If we write

$$F_n(t) = \int_t^{\pi} \sum_{k=0}^{n} r_{n-k} \cos \frac{(k+\frac{1}{2})u}{\sin (u/2)} \, du$$

then

$$I_2 < \frac{1}{2\pi R_n} | \tilde{f}(x+\pi/n) - \tilde{f}(x-\pi/n) | \, F_n(\pi/n)$$

$$+ \frac{1}{2\pi R_n} \int_{\pi/n}^{\pi} \frac{d}{dt} [\tilde{f}(x+t) - \tilde{f}(x-t)] \, F_n(t) \, dt$$

$$= I_{21} + I_{22}.$$

Since $F_n(t) = \displaystyle\int_t^{\pi} \frac{1}{2 \sin^2 \frac{u}{2}} \sum_{k=0}^{n} (r_{n-k} - r_{n-k-1}) \sin (k+1) \, u \, du + o(1).$

Using the second mean value theorem

$$F_n(t) = \frac{1}{2\sin^2\frac{t}{2}} \int_t^{\xi} \sum_{k=0}^{n} (r_{n-k} - r_{n-k-1}) \sin(k+1)\, u\, du$$

$$+ o(1)\ (t \leqslant \xi \leqslant \pi)$$

thus

$$| F_n(t) | < \frac{1}{2\sin^2\frac{t}{2}} \sum_{k=0}^{n} \frac{| r_{n-k} - r_{n-k-1} |}{k+1} + o(1)$$

$$\leqslant K \cdot \frac{1}{t^2} \cdot \frac{r_n}{n} + o(1) \quad \text{by condition (3.3).}$$

Hence $\quad I_{21} = O\left(\frac{1}{\pi \cdot R_n} \cdot \pi/n \frac{r_n}{n} \cdot \frac{n^2}{\pi^2}\right)$, where K is constant

$$= O\left(\frac{r_n}{R_n}\right).$$

By Stieltjes integral we have

$$I_{22} \leqslant \frac{K}{R_n} \int_{\pi/n}^{\pi} | F_n(t) | \, dt,$$

$$= \frac{K}{R_n} \int_{\pi/n}^{\pi} \frac{r_n}{t^2 n}\, dt,$$

$$= O\left(\frac{r_n}{R_n}\right).$$

Combining I_1, I_{21} and I_{22} we have

$$\| f(x) - t_n^{p \cdot q}(f, x) \| = O\left(\frac{r_n}{R_n}\right)$$

which completes the proof of the theorem

Zamanski (1949) determined a saturation class by considering $(C, 1)$ means of the Fourier series $f(x)$. Sunouchi and Watari (1958-59) and Sunouchi (1960; 1961a, b) determined the class of saturation for various methods of summation. (Cesaro Riesz mean, Cesaro Fejer mean) in the theory of Fourier series and Fourier integrals. We have determined the class of saturation for the Nörlund (N, p_n, q_n) means of the Fourier series $f(x)$. The results of Sunouchi and Watari (1958-59) and Zamanski (1949) are the special cases of our result.

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REFERENCES

Sunouchi, G. (1960). On the class of saturation in the theory of approximation. *Tôhoku Math. J.*, **12**, 339-44.

———— (1961a). On the class of saturation in the theory of approximation II. *Tôhoku Math. J.*, **13**, 112-18.

———— (1961b). On the class of saturation in the theory of approximation III. *Tôhoku Math. J.*, **13**, 320-28

Sunouchi, G., and Watari (1958-1959). On the determination of approximation of the class of saturation in theory of approximation of function. I : *Proc. Japan Acad.*, **34**, 447-81. II : *Tôhoku Math. J.*, **11**, 480-88.

Zamanski, M. (1949). Classes de saturation des certaines procedes d' app desserey de Fourier des fonction centines. *Ann. Sci. Ecole Normale* (Suppl.), **66**, 19-43.