AN INTEGRAL TRANSFORMATION OF SOME CLASSES OF REGULAR UNIVALENT FUNCTIONS

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Let S be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and univalent in $D = \{z : |z| < 1\}$. In this paper we study the functions $\mathcal{F}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} \times \int_0^z \frac{f(te^{i\mu}) - f(te^{i\psi})}{t} dt$ where $\mu \neq \psi$, $0 \leqslant \mu, \psi < 2\mu$ and $f(z)$ belonging to some subclasses of S.

§1. Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in D. Let $C(\alpha)$, $S^*(\alpha)$, $K(\alpha)$ be respectively the subclasses of S consisting of convex, starlike (with respect to origin), close-to-convex functions of order $\alpha (0 \leqslant \alpha < 1)$ in the unit disc D. $K[\alpha, \tau]$ stands for the class of close-to-convex functions of order α and type $\tau (0 \leqslant \tau < 1)$ Libera (1964). Libera (1965) has shown that if $f(z)$ is a member of $C(0)$ or $S^*(0)$ or $K(0)$, then the function

$$F(z) = \frac{2}{z} \int_0^z f(t)\, dt \qquad \ldots(1.1)$$

is also a member of $C(0)$ or $S^*(0)$ or $K(0)$ respectively. In a paper Singh (1970) studied the properties of functions

$$\mathcal{F}(z) = \tfrac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt \qquad \ldots(1.2)$$

where $\mathcal{F}(z)$ belongs to the classes $C(0)$, $S^*(0)$ and $K(0)$ respectively.

In this paper we shall study the functions

$$\mathcal{F}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} \int_0^z \frac{f(te^{i\mu}) - f(te^{i\psi})}{t}\, dt \qquad \ldots(1.3)$$

where $\mu \neq \psi$, $0 \leqslant \mu$, $\psi < 2\pi$ and $f(z)$ belongs to the classes $C(\alpha)$, $S^*(\alpha)$, $K(\alpha)$ and $K[\alpha, \tau]$ respectively. The representation (1.3) serves as a transformation on above considered classes for a given μ, $\psi \in [0, 2\pi)$ with $\mu \neq \psi$. With the help of transformation (1.3) and for a given convex function or starlike function or close-to-convex function, we will generate sub-classes of $C(\alpha)$ or $S^*(\alpha)$ or $K(\alpha)$ respectively. Moreover the results of Singh (1970) turn out to be special cases of our results.

§2. Let $f(z)$ be a given regular function in D. We define the following classes $S(f)$ and $s(f)$ corresponding to the function $f(z)$ by

$$S(f) = \left\{F_{\mu,\psi}(z) : F_{\mu,\psi}(z) = \frac{f(ze^{i\mu}) - f(ze^{i\psi})}{e^{i\mu} - e^{i\psi}}, \right.$$
$$\left. \text{where } \mu \neq \psi,\ 0 \leqslant \mu,\ \psi < 2\pi \right\} \qquad \ldots(2.1)$$

and

$$s(f) = \left\{\mathcal{F}_{\mu,\psi}(z) : \mathcal{F}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} \int_0^z \frac{f(te^{i\mu}) - f(te^{i\psi})}{t}\, dt \right.$$
$$\left. \text{where } \mu \neq \psi,\ 0 \leqslant \mu,\ \psi < 2\pi \right\}. \qquad \ldots(2.2)$$

We now prove the following results.

Theorem 2.1 — If $f(z)$ is in $C(\alpha)$, $(0 \leqslant \alpha < 1)$, then $S(f) \subset S^*(\alpha)$.

PROOF : Since $f(z)$ is in $C(\alpha)$, we have

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \text{ for } z \in D. \qquad \ldots(2.3)$$

Take

$$A(z) = 1 + \frac{zf''(z)}{f'(z)}$$

then

$$P(z) = \frac{A(z) - \alpha}{1 - \alpha}$$

has properties $P(0) = 1$, $\operatorname{Re} P(z) > 0$ in D. By a well-known result (Nehari 1952) we have

$$| P(z) - b | < b = b(r) \text{ for } | z | \leqslant r, 0 < r < 1 \qquad ...(2.4)$$

where $b(r)$ is some positive real number. Therefore from (2.3) and (2.4), we have

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \alpha - a \right| < a = a(r) \text{ for } | z | \leqslant r,\ 0 < r < 1, \qquad ...(2.5)$$

where $a(r)$ is some positive real number. Let

$$h(z) = 1 + \frac{zf''(z)}{f'(z)} - \alpha - a$$

i.e.,

$$zf''(z) + (1 - \alpha - a)\, f'(z) = h(z)\, f'(z). \qquad ...(2.6)$$

Let z_1, z_2 be any two points in $| z | \leqslant r$ and let L denote the segment $\overline{f(z_1), f(z_2)}$ in the w-plane ($w = f(z)$). Let L^{-1} denote the image of L under $f^{-1}(w)$, then integrating (2.6) along L^{-1} we get

$$\left| \int_{L^{-1}} \{zf''(z) + (1 - \alpha - a)\, f'(z)\}\, dz \right| = \left| \int_{L^{-1}} h(z)\, f'(z)\, dz \right|,$$

or

$$| z_2 f'(z_2) - z_1 f'(z_1) - (\alpha - a)\{f(z_2) - f(z_1)\} | = \left| \int_L h(z)\, dw \right|$$

$$< a\, | f(z_2) - f(z_1) | .$$

Therefore,

$$\left| \frac{z_2 f'(z_2) - z_1 f'(z_1)}{f(z_2) - f(z_1)} - \alpha - a \right| < a \text{ for } | z | \leqslant r. \qquad ...(2.7)$$

Putting $z_2 = ze^{i\mu}$, $z_1 = ze^{i\psi}$, $\mu \neq \psi$; $0 \leqslant \mu, \psi < 2\pi$ in (2.7), we have

$$\operatorname{Re} \left\{ z \frac{e^{i\mu} f'(ze^{i\psi}) - e^{i\mu} f'(ze^{i\psi})}{f(ze^{i\mu}) - f(ze^{i\psi})} \right\} > \alpha$$

for $| z | \leqslant r$.

Or

$$\operatorname{Re} \left[\frac{z F'_{\mu,\psi}(z)}{F_{\mu,\psi}(z)} \right] > \alpha, \ z \in D,$$

where $F_{\mu,\psi}(z) \in S(f)$. Hence $F_{\mu,\psi}(z) \in S^*(\alpha)$. This implies that $S(f) \subset S^*(\alpha)$.

On similar lines we can prove the following result.

Theorem 2.2 — If $f(z)$ is close-to-convex function of order α $(0 \leqslant \alpha < 1)$ in D with respect to $g(z)$, then

$$\mathrm{Re}\left[\frac{F_{\mu,\psi}(z)}{G_{\mu,\psi}(z)}\right] > \alpha;\ z \in D$$

where $F_{\mu,\psi}(z) \in S(f)$, $G_{\mu,\psi}(z) \in S(g)$ and $\mu \neq \psi$, $0 \leqslant \mu, \psi < 2\pi$.

Theorem 2.3 — If $f(z)$ is in $S^*(\alpha)$, then $s(f) \subset S^*(\alpha)$, $0 \leqslant \alpha < 1$.

PROOF: Let $\mathcal{F}_{\mu,\psi}(z) \in s(f)$ where $f(z) \in S^*(\alpha)$. Hence from definition for the class $s(f)$, we have

$$\mathcal{F}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} \int_0^z \frac{f(te^{i\mu}) - f(te^{i\psi})}{t}\, dt,\quad \mu \neq \psi,\ 0 \leqslant \mu,\ \psi < 2\pi,$$

or

$$z\mathcal{F}'_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} [f(ze^{i\mu}) - f(ze^{i\psi})]$$

$$= \frac{1}{e^{i\mu} - e^{i\psi}} [ze^{i\mu}\, \phi'(ze^{i\mu}) - ze^{i\psi}\, \phi'(ze^{i\psi})] \qquad \text{...(2.8)}$$

where $\phi(z)$ is some convex function of order $\alpha(0 \leqslant \alpha < 1)$. Integrating (2.8) from 0 to z, we have

$$\mathcal{F}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} [\phi(ze^{i\mu}) - \phi(ze^{i\psi})].$$

Therefore $\mathcal{F}_{\mu,\psi}(z) \in S^*(\alpha)$ follows from Theorem 2.1. Hence $s(f) \subset S^*(\alpha)$ which proves the Theorem 2.3.

Theorem 2.4 — If $g(z) \in C(\alpha)$ then $s(g) \subset C(\alpha)$, $0 \leqslant \alpha < 1$.

PROOF: Let $\mathcal{G}_{\mu,\psi}(z)$, $\mu \neq \psi$, $0 \leqslant \mu,\ \psi < 2\pi$ be in $s(g)$ where $g(z)$ is a member of $C(\alpha)$. Then from definition for the class $s(g)$ we have

$$\mathcal{G}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} \int_0^z \frac{g(te^{i\mu}) - g(te^{i\psi})}{t}\, dt,$$

or

$$z\,\mathcal{G}'_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} [g(ze^{i\mu}) - g(ze^{i\psi})].$$

Therefore, from Theorem 2.1, it follows that $z\,\mathcal{G}'_{\mu,\psi}(z) \in S^*(\alpha)$, which in turn implies that $\mathcal{G}_{\mu,\psi}(z) \in C(\alpha)$. Hence $s(f) \subset C(\alpha)$ which proves the theorem.

Theorem 2.5 — If $f(z)$ is close-to-convex function of order α $(0 \leqslant \alpha < 1)$ in D with respect to convex function $g(z)$, then each $\mathcal{F}_{\mu,\psi}(z) \in s(f)$, $\mu \neq \psi$, $0 \leqslant \mu$, $\psi < 2\pi$, is close-to-convex function of order α in D with respect to the corresponding function $\mathcal{G}_{\mu,\psi}(z) \in s(g)$.

PROOF: The definitions for the classes $s(f)$ and $s(g)$ allow us to write

$$\frac{\mathcal{F}'_{\mu,\psi}(z)}{\mathcal{G}'_{\mu,\psi}(z)} = \frac{f(ze^{i\mu}) - f(ze^{i\psi})}{g(ze^{i\mu}) - g(ze^{i\psi})}$$

or

$$\mathrm{Re}\left\{\frac{\mathcal{F}'_{\mu,\psi}(z)}{\mathcal{G}'_{\mu,\psi}(z)}\right\} = \mathrm{Re}\left\{\frac{f(ze^{i\mu}) - g(ze^{i\psi})}{g(ze^{i\mu}) - g(ze^{i\psi})}\right\}.$$

From Theorem 2.2, it follows that

$$\mathrm{Re}\left\{\frac{\mathcal{F}'_{\mu,\psi}(z)}{\mathcal{G}'_{\mu,\psi}(z)}\right\} > \alpha, \text{ for } z \in D.$$

This completes the proof.

The definition of the class $K[\alpha, \tau]$ when combined with the Theorem 2.5 give rise to the following:

Theorem 2.6 — If $f(z) \in K[\alpha, \tau]$, $0 \leqslant \alpha$, $\tau < 1$, then $s(f) \subset K[\alpha, \tau]$.

Theorem 2.7 — If $f(z) \in S$, then each function in $S(f)$ is close-to-convex with respect to some function $\mathcal{G}_{\mu,\psi}(z)$ $\mu \neq \psi$, $0 \leqslant \mu$, $\psi < 2\pi$ in $s(f)$, for $|z| < r_1$ where $r_1 > r_0$ and $0.80 < r_0 < 0.81$.

PROOF: Krzyż (1962) has shown that the radius of close-to-convexity of each $f(z) \in S$ is greater than or equal to r_0 where $0.80 < r_0 < 0.81$. Thus it is clear from Theorem 2.5 that each function

$$\mathcal{F}_{\mu,\psi}(z) = \frac{1}{e^{i\mu} - e^{i\psi}} \int_0^z \frac{f(te^{i\mu}) - f(te^{i\psi})}{t}\, dt, \ \mu \neq \psi, \ 0 \leqslant \mu, \ \psi < 2\pi$$

in $s(f)$ is also close-to-convex function with respect to some function $\mathcal{G}_{\mu,\psi}(z) \in s(f)$ for $|z| < r_1$, $r_1 \geqslant r_0$.

The following results of Singh (1970) follow as corollaries by putting $\mu = 0$, $\psi = \pi$ and $\alpha = 0$ in Theorems 2.3, 2.4, 2.5 and 2.7.

Corollary 1 (*Singh* 1970) — If $f(z) \in S^*(0)$ then $\mathcal{F}(z)$ defined as

$$\mathcal{F}(z) = \tfrac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt$$

is also in $S^*(0)$.

Corollary 2 (*Singh* 1970) — If $g(z) \in C(0)$ then $\mathcal{F}(z)$ defined as

$$\mathcal{F}(z) = \tfrac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt$$

is also in $C(0)$.

Corollary 3 (*Singh* 1970) — If $f(z)$ is close-to-convex with respect to the convex function $g(z)$, then

$$\mathcal{F}(z) = \tfrac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt$$

is close-to-convex with respect to the convex function

$$\mathcal{G}(z) = \tfrac{1}{2} \int_0^z \frac{g(t) - g(-t)}{t}\, dt$$

Corollary 4 (*Singh* 1970) — If $f(z) \in S$, then

$$\mathcal{F}(z) = \tfrac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt$$

is close-to-covex with respect to some function $G(z) \in C$ for $|z| < r_1$, where $r_1 \geqslant r_0$, $.80 < r_0 < .81$.

Remark : From Theorem 2.4 (respectively, Theorems 2.3, 2.5), we notice that a given convex function (respectively, starlike function, close-to-convex function) generate a subclass of convex functions (respectively, starlike functions, close-to-convex functions). Compare it with the corresponding results of Singh (1970). In this case the sub-classes so generated consist of singletons.

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