# PSEUDO $H$-PROJECTIVE CHANGES IN $GF$-STRUCTURE

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In this paper we have studied pseudo $H$-projective changes in $V_n$ equipped with $GF$-structure (Duggal 1971) and have obtained pseudo $H$-projective curvature tensor.

## 1. Introduction

Let us consider a differentiable manifold $V_n$ of class $C^\infty$. It is equipped with $GF$-structure if there exists on $V_n$ a vector valued function $F$ of class $C^\infty$ such that

$$\bar{\bar{X}} = a^2 X \qquad \text{...(1.1)}$$

for any arbitrary vector field $X$ where $a$ is a complex constant and $\bar{X} = F(X)$.

In the sequel arbitrary vector fields are denoted by $X, Y, Z, \ldots$ etc.

*Definition* — A connection $D$ in $V_n$ equipped with $GF$-structure is called an $F$-connection if

$$(D_X F)(Y) = 0. \qquad \text{...(1.2)}$$

An $F$-connection $D$ is called pseudo half-symmetric if its torsion tensor $S(X, Y)$ satisfies

$$a^2 S(X, Y) + S(\bar{X}, \bar{Y}) + \overline{S(\bar{X}, Y)} + \overline{S(X, \bar{Y})} = 0 \qquad \text{...(1.3)}$$

and it is called pseudo semi-symmetric if its torsion tensor $S(X, Y)$ satisfies

$$n a^2 S(X, Y) = a^2 S(X)\, Y - a^2 S(Y)\, X - S(\bar{X})\, \bar{Y} + S(\bar{Y})\, \bar{X} \qquad \text{...(1.4)}$$

where

$$S(X) = (C_1^1\, S)(X).$$

## 2. Pseudo $H$-Projective Changes

In an $GF$-structure with an $F$-connection $D$ we consider a curve $\sigma(t)$. A curve $\sigma(t)$ is called pseudo holomorphically planar if it satisfies

$$D_T T = \alpha(t)\, T + \beta(t)\, \bar{T}$$

where $\alpha(t)$ and $\beta(t)$ are certain functions of the parameter $t$. $T$ is a vector parallel to the tangent vector field of the curve. If $F$-connections $D$ and $D^*$ have all pseudo holomorphically planar curves in common, they are called pseudo $H$-projectively related to each other.

*Theorem* 2.1 — In $V_n$ equipped with $GF$-structure the two pseudo half-symmetric $F$-connections $D$ and $D^*$ are pseudo $H$-projectively related to each other if and only if

$$2a^2D^*_X Y = 2a^2D_X Y + a^2P(X)\,Y + a^2P(Y)\,X + P(\bar{X})\,\bar{Y}$$

$$+ P(\bar{Y})\,\bar{X} + 2a^2Q(X, Y) - 2Q(\bar{X})\,\bar{Y} \qquad ...(2.1)$$

holds for certain 1-forms $P(X)$ and $Q(X)$.

PROOF : If (2.1) holds, then both connections $D$ and $D^*$ have all pseudo holomorphically planar curves in common. Therefore by definition the connections are pseudo $H$-projectively related.

Conversely we suppose that $D$ and $D^*$ are pseudo $H$-projectively related. Let us put

$$2A(X, Y) = D^*_X Y - D_Y X \qquad ...(2.2)$$

then

$$A(V, V) = aV + b\bar{V} \qquad ...(2.3)$$

must hold for any vector field $V$ at any point where $a$ and $b$ depend on both $V$ and the point. Therefore, we have

$$2A(X, Y) = u(X)\,Y + u(Y)\,X + v(X)\,\bar{Y} + v(Y)\,\bar{X} + 2P(X, Y) \qquad ...(2.4)$$

for certain 1-forms $u(X)$ and $v(X)$ and

$$2P(X, Y) = A(X, Y) - A(Y, X). \qquad ...(2.5)$$

Since both $D$ and $D^*$ are pseudo half-symmetric $F$-connections we have

$$\left.\begin{aligned} a^2A(X, Y) &= \overline{A(X, \bar{Y})} \\ a^2P(X, Y) + P(\bar{X}, \bar{Y}) + \overline{P(X, \bar{Y})} + \overline{P(\bar{X}, Y)} &= 0. \end{aligned}\right\} \qquad ...(2.6)$$

We have the identity

$$[a^2P(X, Y) - \overline{P(X, \bar{Y})} + P(\bar{X}, \bar{Y}) - \overline{P(\bar{X}, Y)}]$$

$$+ [a^2P(X, Y) + \overline{P(X, \bar{Y})} - \overline{P(\bar{X}, \bar{Y})} - \overline{P(\bar{X}, Y)}] = 2[a^2P(X, Y) - \overline{P(\bar{X}, Y)}].$$

$$...(2.7)$$

Since $P(X, Y)$ is skew-symmetric, (2.7) reduces to

$$[a^2P(X, Y) - P(\bar{X}, \bar{Y}) + \overline{P(X, \bar{Y})} - \overline{P(\bar{X}, Y)} = 2[a^2P(Y, X) - \overline{P(Y, \bar{X})}]$$
$$- [a^2P(X, Y) - \overline{P(X, \bar{Y})} + P(\bar{X}, \bar{Y}) - \overline{P(\bar{X}, Y)}]. \quad ...(2.8)$$

From (2.6) and (2.8) we have

$$4a^2P(X, Y) = 2[a^2P(X, Y) - \overline{P(X, \bar{Y})}] - 2[a^2P(Y, X) - \overline{P(Y, \bar{X})}]$$
$$- [a^2P(X, Y) - \overline{P(X, \bar{Y})} + P(\bar{X}, \bar{Y}) - \overline{P(\bar{X}, Y)}]. \quad ...(2.9)$$

From (2.4) and (2.6) we have,

$$2[a^2P(X, Y) - \overline{P(X, \bar{Y})}] = - [a^2(u(Y) - v(\bar{Y}))X + (a^2v(Y) - u(\bar{Y}))\bar{X}]. \quad ...(2.10)$$

Substituting (2.10) in (2.9) we get

$$4a^2P(X, Y) = a^2u(X)Y - a^2u(Y)X - u(\bar{X})\bar{Y} + u(\bar{Y})\bar{X}$$
$$- a^2v(\bar{X})Y + a^2v(\bar{Y})X + a^2v(X)\bar{Y} - a^2v(Y)\bar{X}. \quad ...(2.11)$$

Substituting (2.11) in (2.4) and putting

$$P(X) = u(X) + v(\bar{X}) \text{ and } Q(X) = u(X) - v(\bar{X})$$

we find

$$4a^2A(X, Y) = a^2P(X)Y + a^2P(Y)X + P(\bar{X})\bar{Y} + P(\bar{Y})\bar{X}$$
$$+ 2a^2Q(X, Y) - 2Q(\bar{X})\bar{Y}. \quad ...(2.12)$$

From (2.12) and (2.2) we get (2.1). Hence the theorem.

*Theorem* 2.2 — In $V_n$ equipped with $GF$-structure two symmetric $F$-connections $D$ and $D^*$ are pseudo $H$-projectively related to each other if and only if

$$2a^2D^*_X Y = 2a^2D_XY + a^2P(X)Y + a^2P(Y)X + P(\bar{X})\bar{Y} + P(\bar{Y})\bar{X} \quad ...(2.13)$$

holds for certain 1-form $P(X)$.

Proof is obvious from Theorem 2.1.

*Definition* 2.1 — Let $D$ and $D^*$ be two pseudo half-symmetric $F$-connections satisfying (2.1) for certain 1-forms $P(X)$ and $Q(X)$. Then the correspondence $D \rightarrow D^*$ is called a pseudo holomorphically projective transformation of $D$ or shortly a pseudo $H$-projective transformation of $D$.

*Theorem* 2.3 — In $V_n$ equipped with $GF$-structure the tensor

$$a^2S(X, Y) - \frac{1}{n}[a^2S(X)\,Y - a^2S(Y)\,X - S(\bar{X})\,\bar{Y} + S(\bar{Y})\,\bar{X}] \quad ...(2.14)$$

where $S(X, Y)$ is the torsion tensor of a pseudo half-symmetric $F$-connection $D$, is invariant under pseudo $H$-projective transformation of $D$.

PROOF : Let $S^*(X, Y)$ is the torsion tensor of pseudo half-symmetric $F$-connection $D^*$. Then from (2.1) we have

$$a^2S^*(X, Y) = a^2S(X, Y) + a^2Q(X)\,Y - a^2Q(Y)\,X - Q(\bar{X})\,\bar{Y} + Q(\bar{Y})\bar{X}. \quad ...(2.15)$$

On contraction $C_2^1$ of the above equation and by virtue of the fact $C_1^1\,F(X) = 0$, we have

$$Q(X) = \frac{1}{n}(S^*(X) - S(X)). \quad ...(2.16)$$

Substituting (2.16) in (2.15) we obtain

$$a^2S^*(X, Y) - \frac{1}{n}[a^2S^*(X)\,Y - a^2S^*(Y)\,X + S^*(\bar{X})\,\bar{Y} + S^*(\bar{Y})\,\bar{X}$$

$$= a^2S(X, Y) - \frac{1}{n}[a^2S(X)\,Y - a^2S(Y)\,X - S(\bar{X})\,\bar{Y} + S(\bar{Y})\,\bar{X}]$$

which yields the invariancy of (2.14).

*Theorem* 2.4 — In order that a pseudo half-symmetric $F$-connection in $V_n$ equipped with a $GF$-structure be pseudo semi-symmetric, it is necessary and sufficient and it be pseudo $H$-projectively related by a symmetric $F$-connection.

PROOF : Suppose that a pseudo half-symmetric $F$-connection $D$ be pseudo semi-symmetric. Let us consider a pseudo $H$-projectively related $F$-connection $E$ defined by

$$2a^2E_XY = 2a^2D_XY - \frac{1}{n}[a^2S(X)\,Y - a^2S(Y)\,X - S(\bar{X})\,\bar{Y} + S(\bar{Y})\,\bar{X}] \quad ...(2.17)$$

where $S$ is the torsion tensor of $D$ and $S(X) = (C_2^1\,S)\,(X)$. Let $T(X, Y)$ be the torsion tensor of $E$. Then

$$a^2T(X, Y) = a^2S(X, Y) - \frac{1}{n}[S(X)\,Y - S(Y)\,X + S(\bar{Y})\,\bar{X} - S(\bar{X})\,\bar{Y})] \quad ...(2.18)$$

which implies that $T(X, Y)$ is zero on account of the pseudo semi-symmetric condition (1.4) of $D$.

Conversely if $D$ is pseudo $H$-projectively related to a symmetric $F$-connection, then from Theorem 2.3 we get (1.4) which shows that $D$ is pseudo semi-symmetric.

## 3. Pseudo $H$-Projective Flatness and Pseudo $H$-Projective Curvature Tensor

Let $D$ be a pseudo half-symmetric $F$-connection in $V_n$ equipped with $GF$-structure. We assume that for any point in the space there exists at least a neighbourhood of the point in which $D$ is pseudo $H$-projectively related to a flat $F$-connection. Then the pseudo half-symmetric $F$-connection is said to be pseudo $H$-projectively flat.

*Theorem* 3.1 — In order that a pseudo half-symmetric $F$-connection $D$ be pseudo $H$-projectively flat, it is necessary and sufficient that there exists a pseudo $H$-projectively flat symmetric $F$-connection which is pseudo $H$-projectively related to the given connection.

Proof : Let us assume that a pseudo half-symmetric $F$-connection $D$ be pseudo $H$-projectively flat. Then the connection $E$ defined by (2.17) is necessarily symmetric. By virtue of the Theorem 2.3 we observe that the torsion tensor $T(X, Y)$ given by (2.18) vanishes because of pseudo $H$-projective flatness of $D$. Further the symmetric $F$-connection $E$ is also pseudo $H$-projectively flat.

We define pseudo $H$-projective curvature tensor $P(X, Y, Z)$ of a symmetric $F$-connection $D$ by

$$\begin{aligned} 2a^2P(X, Y, Z) &= 2a^2R(X, Y, Z) + a^2Q(Y, Z)\,X - a^2Q(X, Z)\,Y \\ &\quad - a^2\,[Q(X, Y) - Q(Y, X)]\,Z + Q(Y, \bar{Z})\,\bar{X} \\ &\quad - Q(X, \bar{Z})\,\bar{Y} - Q(X, \bar{Y})\,\bar{Z} + Q(Y, \bar{X})\,\bar{Z} \qquad \ldots(3.1) \end{aligned}$$

where $Q(X, Y)$ is defined by

$$Q(X, Y) = -\frac{2}{n+2}\left\{\text{Ric}\,(X, Y) + \frac{2}{n-2}\,(O\,(\text{Ric}\,(X, Y) + \text{Ric}\,(Y, X))\right\}$$

where $R(X, Y, Z)$ and Ric $(X, Y)$ are the curvature tensor and Ricci tensor of $V_n$ equipped with the $GF$-structure.

*Theorem* 4.2 — In $V_n$ equipped with $GF$-structure any two symmetric $F$-connections which are pseudo $H$-projectively related to each other, have the pseudo $H$-projective curvature tensor in common.

Proof : Let $D$ and $D^*$ be two pseudo $H$-projectively related symmetric $F$-connections. Let $R(X, Y, Z)$ and $^*R(X, Y, Z)$ be their curvature tensors. Then by (2.13) we have

$$
\begin{aligned}
4a^2\, {}^*R(X, Y, Z) = {} & 4a^2 R(X, Y, Z) \\
& + [a^2 P(Z)\, P(Y) + P(\bar{Y})\, P(\bar{Z}) - 2a^2 (D_Y P)(Z)] \\
& - [a^2 P(X)\, P(Z) + P(\bar{X})\, P(\bar{Z}) - 2a^2 (D_X P)(Z)]\, Y \\
& + 2a^2 [(D_X P)(Y) - (D_Y P)(X)]\, Z \\
& + [P(Y)\, P(\bar{Z}) + P(\bar{Y})\, P(Z) - 2(D_Y P)(\bar{Z})]\, \bar{X} \\
& - [P(X)\, P(\bar{Z}) + P(\bar{X})\, P(Z) - 2(D_X P)(\bar{Z})]\, \bar{Y} \\
& + 2\, [(D_X P)(\bar{Y}) - (D_Y P)(\bar{X})]\, \bar{Z}. \qquad \ldots(3.2)
\end{aligned}
$$

Putting

$$
- 2a^2 M(X, Y) = - 2a^2 (D_X P)(Y) + a^2 P(X)\, P(Y) + P(\bar{X})\, P(\bar{Y}) \qquad \ldots(3.3)
$$

in (3.2) we get

$$
\begin{aligned}
2a^2\, {}^*R(X, Y, Z) = {} & 2a^2\, {}^*R(X, Y, Z) - a^2 M(Y, Z)\, X + a^2 M(X, Z)\, Y \\
& + a^2 [M(X, Y) - M(Y, X)]\, Z - M(Y, \bar{Z})\, \bar{X} \\
& + M(X, \bar{Z})\, \bar{Y} + [M(X, \bar{Y}) - M(Y, \bar{X})]\, \bar{Z}. \qquad \ldots(3.4)
\end{aligned}
$$

Contracting (3.4) with respect to $X$, we get

$$
\begin{aligned}
2a^2\, {}^*\mathrm{Ric}\,(Y, Z) = {} & 2a^2\, \mathrm{Ric}\,(Y, Z) - a^2 (n + 2)\, M(Y, Z) + a^2 M(Y, Z) \\
& + a^2 M(Z, Y) + M(\bar{Y}, \bar{Z}) + M(\bar{Z}, \bar{Y}) \qquad \ldots(3.5)
\end{aligned}
$$

which yields

$$
\begin{aligned}
& 2a^2\, [{}^*\mathrm{Ric}\,(Y, Z) - {}^*\mathrm{Ric}\,(Z, Y)] - 2a^2\, [\mathrm{Ric}\,(Y, Z) - \mathrm{Ric}\,(Z, Y)] \\
& \qquad = - a^2 (n + 2)\, [M(Y, Z) - M(Z, Y)]. \qquad \ldots(3.6)
\end{aligned}
$$

(3.6) gives

$$
\begin{aligned}
OM(Y, Z) - OM(Z, Y) = {} & - \frac{2}{n + 2} [(O\, {}^*\mathrm{Ric}\,(Y, Z) - O\, {}^*\mathrm{Ric}\,(Z, Y)) \\
& - (O\, \mathrm{Ric}\,(Y, Z) - O\, \mathrm{Ric}\,(Z, Y))] \qquad \ldots(3.7)
\end{aligned}
$$

where

$$
2a^2 OM(Y, Z) = a^2 M(Y, Z) + M(\bar{Y}, \bar{Z}).
$$

From (3.5) and (3.7) we get

$$
\begin{aligned}
& 2a^2\, [{}^*\mathrm{Ric}\,(Y, Z) - \mathrm{Ric}\,(Y, Z)] - \frac{4a^2}{n - 2} [(O\, {}^*\mathrm{Ric}\,(Y, Z) - O\, {}^*R(Z, Y)) \\
& \qquad - (O\, \mathrm{Ric}(Y, Z) - O\, \mathrm{Ric}\,(Z, Y))] \\
& \qquad = - a^2 (n + 2)\, M(Y, Z) + 4a^2 OM(Y, Z). \qquad \ldots(3.8)
\end{aligned}
$$

From (3.5) by operating $O$ we get,

$$2(O\,{}^*\text{Ric}(Y,Z) - O\,\text{Ric}(Y,Z) - \frac{4}{n+2}[(O\,{}^*\text{Ric}(Y,Z) - O\,{}^*\text{Ric}(Z,Y))$$

$$- (O\,\text{Ric}(Y,Z) - O\,\text{Ric}(Z,Y))] = -(n-2)\,OM(Y,Z) \quad ...(3.9)$$

which yields

$$OM(Y,Z) + OM(Z,Y) = -\frac{2}{n-2}[O({}^*\text{Ric}(Y,Z) + {}^*\text{Ric}(Z,Y))$$

$$- O(\text{Ric}(Y,Z) + \text{Ric}(Z,Y))]. \quad ...(3.10)$$

Substituting (3.10) in (3.5) we get

$$M(Y,Z) = -\frac{2}{n+2}\Big[{}^*\text{Ric}(Y,Z) - \text{Ric}(Y,Z) + \frac{2}{n-2}\Big\{O({}^*\text{Ric}(Y,Z)$$

$$+ {}^*\text{Ric}(Z,Y)) - O(\text{Ric}(Y,Z) + \text{Ric}(Z,Y))\Big\}\Big]$$

Hence

$$M(Y,Z) = {}^*Q(Y,Z) - Q(Y,Z) \quad ...(3.11)$$

where

$$Q(Y,Z) = -\frac{2}{n+2}\Big[\text{Ric}(Y,Z) + \frac{2}{n-2}\Big\{O(\text{Ric}(Y,Z) + \text{Ric}(Z,Y)\Big\}\Big]. \quad ...(3.12)$$

Putting (3.11) in (3.4) we get with the help of (3.1)

$$2a^2\,{}^*P(X,Y,Z) = 2a^2\,P(X,Y,Z) \quad ...(3.13)$$

where ${}^*P(X, Y, Z)$ is pseudo $H$-projective curvature of $D^*$. Hence the theorem.

*Theorem* 4.3 — In $V_n$ $(n > 2)$ equipped with $GF$-structure, a symmetric $F$-connection is pseudo $H$-projectively flat if and only if its pseudo $H$-projective curvature tensor vanishes identically.

PROOF: When a symmetric $F$-connection $D$ is pseudo $H$-projectively flat, $P(X, Y, Z)$ obviously vanishes. Conversely if we suppose that $P(X, Y, Z)$ vanishes identically, we have from (3.1)

$$2a^2R(X,Y,Z) = -a^2Q(Y,Z)\,X + a^2Q(X,Z)\,Y$$

$$+ a^2[Q(X,Y) - Q(Y,X)]\,Z - Q(Y,\bar{Z})\,\bar{X} + Q(X,\bar{Z})\,\bar{Y}$$

$$+ Q(X,\bar{Y})\,\bar{Z} - Q(Y,\bar{X})\,\bar{Z}. \quad ...(3.14)$$

By (3.4) in order to prove that the $F$-connection $D$ is pseudo $H$-projectively flat, it is sufficient to show that in any neighbourhood there exist a local 1-form $P(X)$ such that

$$2a^2(D_XP)(Y) = 2a^2Q(X, Y) - a^2P(X)P(Y) - P(\bar{X})P(\bar{Y}). \quad ...(3.15)$$

The pseudo integrability condition of the differential equation (3.15) with its application is given by

$$\begin{aligned} -2a^2P(R(X, Y, Z) &= 2a^2(D_XQ)(Y, Z) - 2a^2(D_YQ)(X, Z) \\ &- a^2[Q(X, Y) - Q(Y, X)]P(Z) - P(\bar{Z})[Q(X, \bar{Y}) - Q(Y, \bar{X})] \\ &- a^2P(Y)Q(X, Z) + a^2P(X)Q(Y, Z) - P(\bar{Y})Q(X, \bar{Z}) + P(\bar{X})Q(Y, \bar{Z}) \end{aligned} \quad ...(3.16)$$

which yields

$$(D_XQ)(Y, Z) = (D_YQ)(X, Z). \quad ...(3.17)$$

Now we shall show that the condition (3.14) implies the pseudo integrability condition (3.14) of (3.15). By means of Bianchi's identity (3.14) gives

$$\begin{aligned} & a^2[-(D_UQ)(Y, Z) + (D_YQ)(U, Z)]X + a^2[(D_UQ)(X, Z) - (D_XQ)(U, Z)]Y \\ & + a^2[(D_UQ)(X, Y) - (D_UQ)(Y, X) + (D_XQ)(Y, U) - (D_XQ)(U, Y) \\ & + (D_YQ)(U, X) - (D_YQ)(X, U)]Z + a^2[(D_XQ)(Y, Z) - (D_YQ)(X, Z)]U \\ & + [-(D_UQ)(Y, \bar{Z}) + (D_YQ)(U, \bar{Z})]\bar{X} + [(D_UQ)(X, \bar{Z}) - (D_XQ)(U, \bar{Z})]\bar{Y} \\ & + [(P_UQ)(X, \bar{Y}) - (D_UQ)(Y, \bar{X}) + (D_XQ)(Y, \bar{U}) - (D_XQ)(U, \bar{Y}) \\ & + (D_YQ)(U, \bar{X}) - (D_YQ)(X, \bar{U})]\bar{Z} + [(D_XQ)(Y, \bar{Z}) - (D_YQ)(X, \bar{Z})]\bar{U} = 0. \end{aligned} \quad ...(3.18)$$

(3.18) implies (3.17). Thus the differential equation (3.15) is completely integrable under the condition (3.14). Hence the theorem is proved.

*Remark*: For $a = \pm i$ all the theorems hold for almost complex manifold (Yano 1965) and for $a^2 = 1$ hold for almost product manifold (Sinha 1974).

## References

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