GREEN'S FUNCTIONS FOR INFINITE ELASTIC PLATES CLAMPED ALONG INNER CURVILINEAR EDGES

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Complex variable methods are used to find exact and closed expressions for the deflections of thin infinite slabs subject to isolated loads at arbitrary points, having their outer edges free and clamped along certain curvilinear boundaries. The plates considered are conformally mapped on the area outside the unit circle by means of rational mapping functions. The interesting cases of an infinite plate clamped along the edge of a crescent-like hole or along a cut having the shape of a circular arc are included as special cases.

1. INTRODUCTION

Dean (1953, 1954) and Dean and Harris (1954) used the simple transformations

$$z = 2\zeta/(1 + \zeta^2), \;\; z = a(\zeta + m\zeta^{-1}), \;\; z = \zeta^2 \qquad ...(1)$$

to obtain closed expressions for the deflections of singularly loaded infinite plates bounded internally by and clamped along two semi-infinite straight lines, an ellipse or a parabola. The special case of a singularly loaded infinite plate clamped along a finite straight cut is included in Dean's paper (1954). The mapping function

$$z = c(1 - \zeta)^2/(1 + \zeta)^2 \qquad ...(2)$$

was used by Bassali (1960a) to discuss the case of an infinite plate clamped along one half of the real axis and loaded by an isolated load at any point. Exact expressions in closed forms were given by Bassali and Dawoud (1957) and Bassali (1960b) for the deflections of singularly loaded infinite perforated plates which are conformally mapped on the region outside the unit circle γ in the ζ-plane by the mapping functions

$$z = c(\zeta + m\zeta^{-p}), \;\; |\, m \,| \leqslant 1/p \qquad ...(3)$$

$$z = c\zeta/(1 + m\zeta^{-p}), \;\; |\, m \,| \leqslant 1/(p + 1) \qquad ...(4)$$

where m and p are real parameters, p being a positive integer. For $p = 1$ the hole obtained by using (3) is an ellipse whose semi-axes are $(1 + m)c$ and $(1 - m)c$, while for $m = -2/p(p+1)$ $(p \geqslant 2)$ we have good approximations to the infinite plane with a cut-out in the form of a regular rectilinear polygon of $p+1$ sides. When $m = 1/p$ the hole is a star-shaped hypocycloid with $p+1$ cusps. The transformation (4) leads to the case of an infinite plane with a notch having the shape of a regular curvilinear polygon with p sides and p rounded vertices which become cusps when $m = 1/(p+1)$. For $m = 2/p(p-1)$ $(p \geqslant 3)$ the hole becomes an approximately regular rectilinear polygon while for $p = 1$, $m \leqslant \frac{1}{2}$ it is bounded by the inverse of an elliptic limacon with respect to its centre (Bassali 1960b, p. 111).

In this paper the methods of complex function theory are used to derive exact and closed expressions for the small transverse displacements due to a normal isolated force acting at an arbitrary point of a thin infinite perforated plate conformally mapped on the domain outside γ by one of the following rational functions :

$$z = c(\zeta + m\zeta^{-1})/(1 - n\zeta^{-1}) \qquad \ldots(5)$$

$$z = c(\zeta + m\zeta^{-1})/(1 + n\zeta^{-2}) \qquad \ldots(6)$$

where m and n are real parameters restricted such that $z'(\zeta)$ does not vanish or become infinite outside γ. The cases of an infinite plate clamped along the boundary of a hole having approximately the shape of a crescent or clamped along a cut with the shape of a circular arc are included as particular cases of (5). Several previously known solutions appear as special cases of the deflections given here. Holes corresponding to certain combinations of the parameters m and n are sketched.

2. Basic Equations

Let $z = x + iy = re^{i\theta}$ be the complex variable of any point in the mid-plane of an infinite plate with an outer free edge and an inner clamped boundary Γ, the plate being normally loaded by a concentrated force F at an arbitrary point $Q_0(z_0)$ outside Γ. Let

$$z = z(\zeta),\ \zeta = \xi + i\eta = \rho e^{i\Psi},\ z'(\zeta) \neq 0, \infty \text{ for } \rho > 1 \qquad \ldots(7)$$

map the infinite region outside Γ conformally on the region outside γ. If ζ_0 corresponds to z_0 it was shown (Bassali 1960b) that the deflection w, measured positively upwards, is given by

$$w = \bar{z}\,\Omega(z) + z\,\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z}) \qquad \ldots(8)$$

where the complex potentials $\Omega(z)$ and $\omega(z)$ take the forms

$$\Omega(z) = \frac{F}{16\pi D}\left[F(\zeta) - (z - z_0)\log\frac{\zeta - \zeta_0}{1 - \bar{\zeta}_0\zeta}\right] \qquad \ldots(9)$$

$$\omega(z) = \frac{F}{16\pi D}\left[f(\zeta) + K\log\zeta + \bar{z}_0(z - z_0)\log\frac{\zeta - \zeta_0}{1 - \bar{\zeta}_0\zeta}\right] \qquad \ldots(10)$$

D being the flexural rigidity of the plate, K a real constant and $F(\zeta)$, $f(\zeta)$ are regular functions of $\zeta(\,|\,\zeta\,|\, \geqslant 1)$. Substitution from (9) and (10) in (8) yields

$$w = \frac{F}{8\pi D}\Big[\,|\,z - z_0\,|\,^2 \log(\rho_0 R_0'/R_0) + K \log \rho + \text{Re}\Big\{\bar{z}F(\zeta) + f(\zeta)\Big\}\Big] \quad ...(11)$$

where

$$\rho_0 = \,|\,\zeta_0\,|\,,\ R_0 = \,|\,\zeta - \zeta_0\,|\,,\ R_0' = \,|\,\zeta - \zeta_0'\,|\,,\ \bar{\zeta}_0\zeta_0' = 1. \quad ...(12)$$

The boundary conditions $w = 0$, $\partial w/\partial\zeta = 0$ on γ lead to

$$\bar{z}(\sigma^{-1})\,F(\sigma) + z(\sigma)\,\bar{F}(\sigma^{-1}) + f(\sigma) + \bar{f}(\sigma^{-1}) = 0 \quad ...(13)$$

$$K + \sigma\,[z'(\sigma)\,\bar{F}(\sigma^{-1}) + \bar{z}(\sigma^{-1})\,F'(\sigma) + f'(\sigma)] + (\rho_0^2 - 1)\,\frac{z(\sigma) - z_0}{\sigma - \zeta_0}\cdot\frac{\bar{z}(\sigma^{-1}) - \bar{z}_0}{\sigma^{-1} - \bar{\zeta}_0} = 0 \quad ...(14)$$

where $\sigma = e^{i\varphi}$ is any point on γ. With the usual notations (Timoshenko and Woinowsky-Krieger 1959) the conditions for the outer edge to be free are

$$M_r \to 0,\ \ Q_r - \frac{1}{r}\,\frac{\partial M_{r\theta}}{\partial\theta} \to 0 \text{ as } r \to \infty \quad ...(15)$$

where the moments and shears at any point z are given by

$$M_r + M_\theta = 8(1 + \nu)D\ \text{Re}\ \Omega'(z) \quad ...(16a)$$

$$M_r - M_\theta + 2iM_{r\theta} = 4(1 - \nu)D\,[z\,\Omega''(z) + z^2\omega''(z)/r^2] \quad ...(16b)$$

$$Q_r - iQ_\theta = 8Dz\,\Omega''(z)/r. \quad ...(16c)$$

3. First Mapping Function

In this section we consider infinite thin perforated plates mapped on the region outside γ by means of

$$z = c(\zeta + m\zeta^{-1})/(1 - n\zeta^{-1}),\ c > 0,\ |\,n\,| \leqslant 1 \quad ...(17)$$

where m, n are real parameters subject to the conditions that $z'(\zeta)$ does not vanish or become infinite outside γ. The parametric equations of the inner clamped edge Γ are

$$\left.\begin{aligned} \frac{x}{c} &= \frac{(1 + m)\cos\psi - n\cos 2\psi - mn}{1 + n^2 - 2n\cos\psi} \\ \frac{y}{c} &= \frac{(1 - m)\sin\psi - n\sin 2\psi}{1 + n^2 - 2n\cos\psi}. \end{aligned}\right\} \quad ...(18)$$

The boundaries Γ corresponding to various combinations of m and n are shown in Figs. 1 and 2. For values of m near -1 the edge Γ resembles the shape of a crescent (see Fig. 3). It is shown (Elsirafy 1968) that the family of curves Γ can be obtained either by inverting an elliptic limacon with respect to an internal point on the axis of symmetry or by inverting an ellipse with respect to an external point on

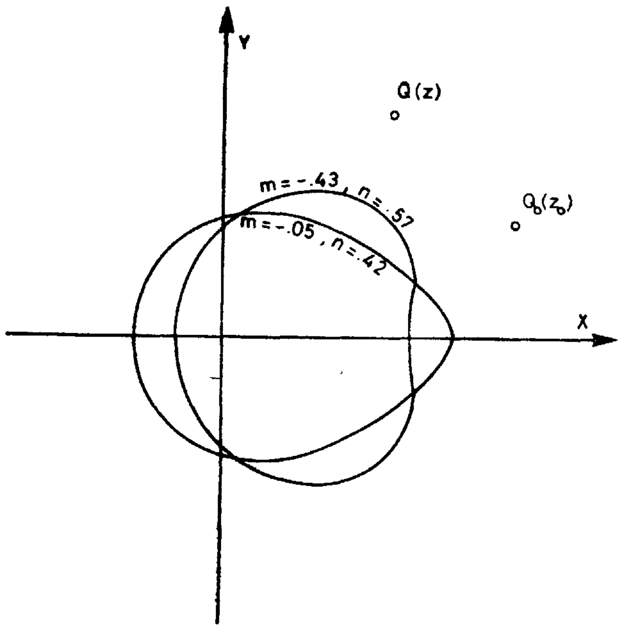

FIG. 1. $z = c(\zeta + m\zeta^{-1})/(1 - n\zeta^{-1})$. Two uniaxially symmetric holes.

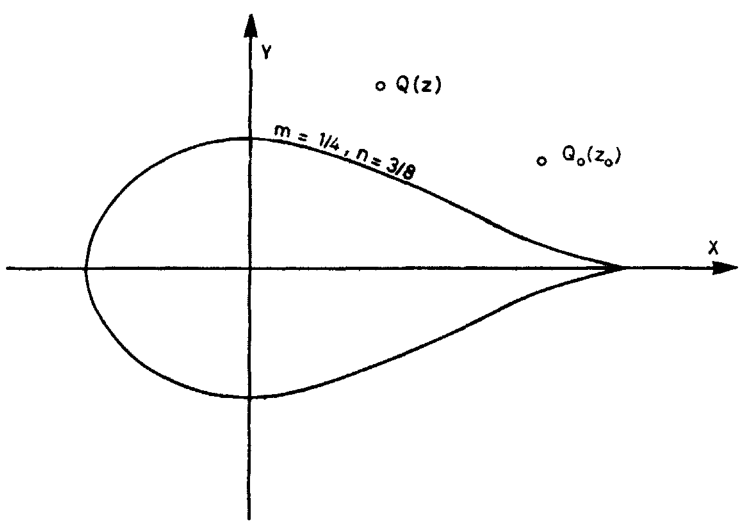

FIG. 2. $z = c(\zeta + m\zeta^{-1})/(1 - n\zeta^{-1})$. A hole with a cusp.

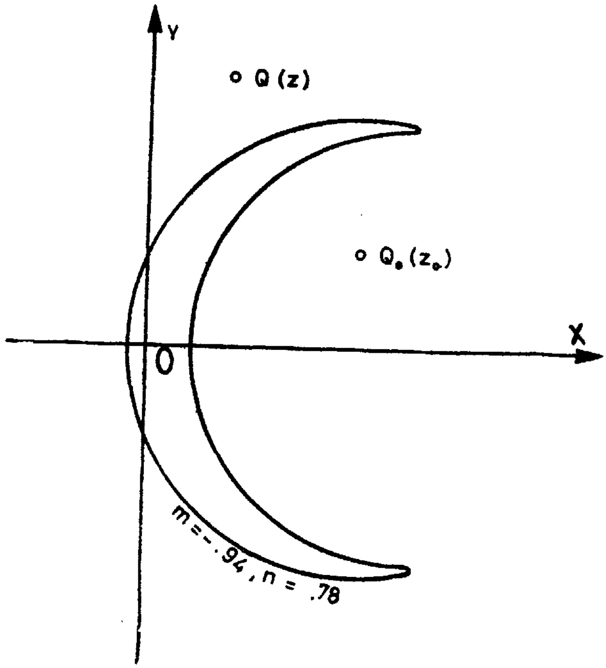

FIG. 3. $z = c(\zeta + m\zeta^{-1})/(1 - n\zeta^{-1})$. A crescent-like hole.

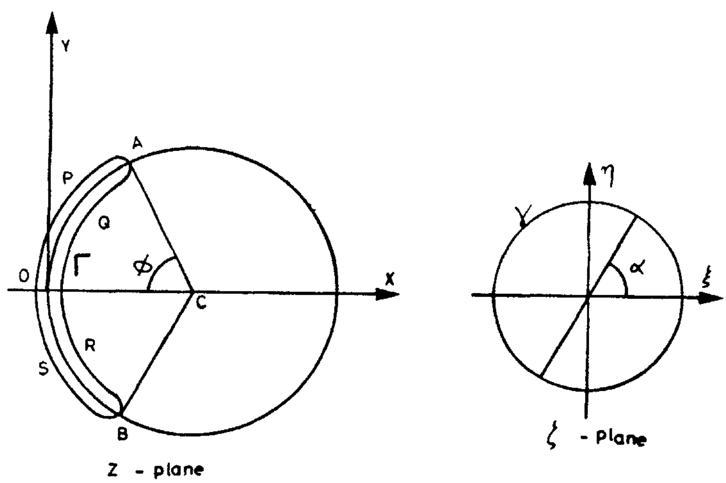

FIG. 4. $z = c(\zeta - \zeta^{-1})/(1 - n\zeta^{-1})$ $(n = \frac{1}{2})$. A cut with the shape of a circular arc.

an axis of symmetry. This latter representation is interesting in as much as it leads to the case in which the inner clamped edge Γ degenerates into a circular arc taken twice (a circular cut). This arises when the ellipse to be inverted degenerates into a straight segment taken twice and the centre of inversion lies on the straight line bisecting this segment at right angles. It will now be shown that the parameter m in this case takes the special value -1. Putting $m = -1$ in (17) and (18) gives the mapping function

$$z = c(\zeta - \zeta^{-1})/(1 - n\zeta^{-1}), \ |n| \leqslant 1 \qquad \text{...(19)}$$

and the parametric equations

$$x = \frac{2cn \sin^2 \psi}{1 + n^2 - 2n \cos \psi}, \ y = \frac{2c \sin \psi(1 - n \cos \psi)}{1 + n^2 - 2n \cos \psi}. \qquad \text{...(20)}$$

The eliminations of ψ leads to the circle

$$(x - c/n)^2 + y^2 = (c/n)^2. \qquad \text{...(21)}$$

As ψ varies from $\psi = 0$ to $\psi = \alpha = \cos^{-1} n$, x varies from $x = 0$ to $x_{max} = 2cn$ and z moves from O to A (see Fig. 4). As ψ increases from α to π the point z in the z-plane returns from A to O. When ψ increases from π to $\pi + \alpha$ then z moves along the circular arc from O to B. As ψ increases from $\pi + \alpha$ to 2π the point z turns back from B to 0. Thus the circular cut OPAQORBSO is traversed by the point z as ζ moves once round γ in the ζ-plane. The circular cut subtends at the centre C of the circle (21) an angle $2\phi = 2 \cos^{-1} (1 - 2n^2)$. For $n = \frac{1}{2}$, $\phi = \pi/3$ while for $n = \sqrt{2}/2$ we have the case of a semi-circular cut. For $n = \sqrt{3}/2$, $\phi = 2\pi/3$ and as $n \to 1$, $\phi \to \pi$.

We now proceed to determine the two functions $F(\zeta)$ and $f(\zeta)$ appearing in (9) and (10) corresponding to the mapping function (17). These will be assumed in the forms

$$F(\zeta) = \frac{c(a_0 \zeta + a_1 + a_2 \zeta^{-1})}{1 - n\zeta^{-1}}, \ f(\zeta) = \frac{c^2(b_0 + b_1 \zeta^{-1})}{1 - n\zeta^{-1}} \qquad \text{...(22)}$$

where a_0, b_0 are real and a_1, a_2, b_1 are complex constants to be determined. The boundary conditions (15) lead to

$$a_0 = -\log \rho_0. \qquad \text{...(23)}$$

Substituting from (22) in (13), (14) and equating the coefficients of various powers of σ to zero we obtain simultaneous linear equations which give

$$\left.\begin{aligned}
a_1 &= nK/c^2 - \bar{C}_1, \ a_2 = m \log \rho_0 \\
K &= Jc^2 [2(1 + m^2 - 2n^2) \log \rho_0 + (n^2 - 1)C_0 \\
&\quad - n(1 + m)(C_1 + \bar{C}_1)] \\
b_0 &= (1 + m) J [\{1 - m - (3 + m) n^2\} \log \rho_0 + n^2 C_0 \\
&\quad + \tfrac{1}{2} n(1 + n^2)(C_1 + \bar{C}_1)] \\
b_1 &= mC_1 + \bar{C}_1 + n(1 + m) J [\{(1 - m)n^2 \\
&\quad - (1 + m + 2m^2)\} \log \rho_0 + C_0 \\
&\quad + \tfrac{1}{2} n(3 + 2m + n^2)(C_1 + \bar{C}_1)]
\end{aligned}\right\} \qquad \text{...(24)}$$

where

$$\left.\begin{aligned} C_0 &= \left(1 - \rho_0^{-2}\right) \left\{(1 + n^2)\rho_0^2 + m^2 + n^2 \right. \\ &\quad \left. + 2n\left(m - 1 \right)\xi_0 \right\} \Big/ \mid 1 - n\zeta_0^{-1} \mid ^2 \\ C_1 &= \left(1 - \rho_0^{-2}\right) \left(mn - m\zeta_0 + n^2\bar{\zeta}_0 \right. \\ &\quad \left. - n\rho_0^2 \right) \Big/ \mid 1 - n\zeta_0^{-1} \mid ^2 \\ J^{-1} &= 1 - 2(1 + m)n^2 - n^4. \end{aligned}\right\} \qquad \ldots(25)$$

When the values of these constants are introduced in (22) and the resulting expressions are inserted in (11) it is found that

$$qw = \left| \frac{z - z_0}{c} \right|^2 \log \frac{\rho_0 R_0'}{R_0} + 2(1 + m^2 - 2n^2)J \log \rho_0 \log \rho + g(\zeta_0) \log \rho$$
$$+ g(\zeta) \log \rho_0 + \frac{(n^2 + m)(1 - \rho_0^{-2})(1 - \rho^{-2})}{\mid 1 - n\zeta_0^{-1} \mid ^2 \mid 1 - n\zeta^{-1} \mid ^2}$$
$$[\{(1 + m)^2 n^2 + (1 - n^2)^2 \xi_0 \xi - n(1 + m)(1 - n^2)(\xi + \xi_0)\} J - \eta_0 \eta] \qquad \ldots(26)$$

where

$$q = 8\pi D/Fc^2$$
$$g(\zeta) = \frac{1 - \rho^{-2}}{\mid 1 - n\zeta^{-1} \mid ^2} [\{2n(1 + m^2 - 2n^2)\xi + n^4 - m^2 - (1 + m)^2 n^2\} J - \rho^2]. \qquad \ldots(27)$$

We are now in a position to consider several interesting special cases :

(*a*) For $m = -1, n < 1$ we have the mapping function (19) and the infinite plate is clamped along the circular cut AOB of Fig. 4. In this case the deflection (26) reduces to

$$qw = \left| \frac{z - z_0}{c} \right|^2 \log \frac{\rho_0 R_0'}{R_0} + \frac{4 \log \rho_0 \log \rho}{1 + n^2} - \frac{(1 - n^2)(1 - \rho_0^{-2})(1 - \rho^{-2})}{\mid 1 - n\zeta_0^{-1} \mid ^2 \mid 1 - n\zeta^{-1} \mid ^2}$$
$$\times \left(\frac{1 - n^2}{1 + n^2} \xi_0 \xi - \eta_0 \eta \right) - \frac{1 - \rho_0^{-2}}{\mid 1 - n\zeta_0^{-1} \mid ^2} \left(1 + \rho_0^2 - \frac{4n \xi_0}{1 + n^2} \right) \log \rho$$
$$- \frac{1 - \rho^{-2}}{\mid 1 - n\zeta^{-1} \mid ^2} \left(1 + \rho^2 - \frac{4n\xi}{1 + n^2} \right) \log \rho_0. \qquad \ldots(28)$$

The case of a clamped semi-circular cut is obtained by putting $n^2 = \frac{1}{2}$ in (28).

(*b*) For $m = n = 0$ the hole is bounded by the circle $|z| = c$ and (26) reduces to

$$qw = \left|\frac{z - z_0}{c}\right|^2 \log \frac{\rho_0 R_0'}{R_0} + 2\log\rho_0 \log\rho + (1 - \rho_0^2)\log\rho + (1 - \rho^2)\log\rho_0 \qquad \ldots(29)$$

where $\rho = |z/c| > 1$, $\rho_0 = |z_0/c| > 1$. This agrees with eqn. (10) of Symonds (1946, p. 187).

(c) For $m = -n^2$ $(n^2 \leqslant 1)$ the plate is bounded internally by and clamped along the circle $|z - nc| = c$ and again we get (29) where $\rho = |z - nc|/c$, $\rho_0 = |z_0 - nc|/c$. It is also easily verified that letting n tend to ± 1 in (28) leads to (29).

(d) The special value $m = 0$ gives the transformation

$$z = c\zeta/(1 - n\zeta^{-1}). \qquad \ldots(30)$$

In this case the inner edge Γ of the infinite plate is the inverse of an elliptic limacon with respect to its centre and (26) agrees with eqn. (4.29) of Bassali (1960b, p. 111) on noting the difference in notation.

(e) For $n = 0$, $0 < m < 1$ we get the mapping function

$$z = c(\zeta + m\zeta^{-1}) \qquad \ldots(31)$$

and the corresponding formula for w becomes

$$qw = \left|\frac{z - z_0}{c}\right|^2 \log \frac{\rho_0 R_0'}{R_0} + 2(1 + m^2)\log\rho_0 \log\rho$$

$$- (1 - \rho_0^{-2})(\rho_0^2 + m^2)\log\rho - (1 - \rho^{-2})(\rho^2 + m^2)\log\rho_0$$

$$+ m(1 - \rho_0^{-2})(1 - \rho^{-2})(\xi_0\xi - \eta_0\eta) \qquad \ldots(32)$$

which is in agreement with Dean's result (1954) for the elliptic hole. For $n = 0$, $m = 1$ we have the case of an infinite plate clamped along the finite straight segment extending from the point $(-2c, 0)$ to the point $(2c, 0)$ in the z-plane and loaded by the concentrated force at $z_0(\zeta_0)$.

4. Second Mapping Function

The parametric equations of the inner edge Γ corresponding to the mapping function

$$z = c\zeta(1 + m\zeta^{-2})/(1 + n\zeta^{-2}) \qquad \ldots(33)$$

are given by

$$\left.\begin{aligned} \frac{x}{c} &= \frac{(1 + m + mn)\cos\psi + n\cos 3\psi}{1 + n^2 + 2n\cos 2\psi} \\ \frac{y}{c} &= \frac{(1 - m + mn)\sin\psi + n\sin 3\psi}{1 + n^2 + 2n\cos 2\psi}. \end{aligned}\right\} \qquad \ldots(34)$$

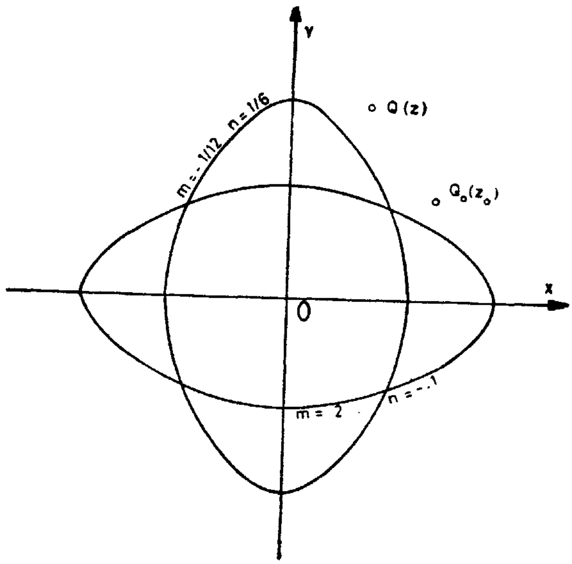

FIG. 5. $z = c(\zeta + m\zeta^{-1})/(1 + n\zeta^{-2})$. Two doubly symmetric holes.

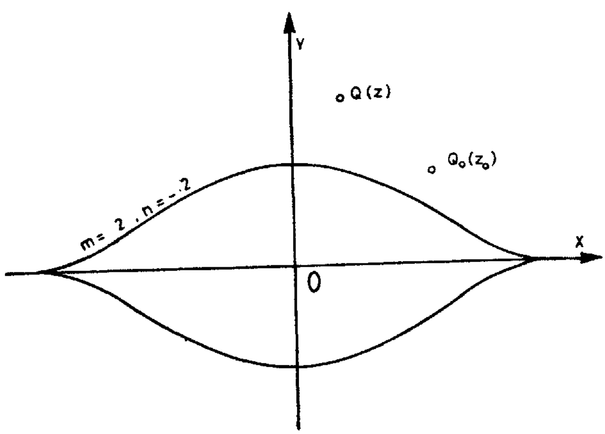

FIG. 6. $z = c(\zeta + m\zeta^{-1})/(1 + n\zeta^{-2})$. A hole with two cusps.

It is easily seen that Γ is symmetrical with respect to both axes. The shapes of three holes corresponding to certain values of m and n are shown in Figs. 5 and 6. The appropriate forms of the two functions $F(\zeta)$ and $f(\zeta)$ are

$$F(\zeta) = \frac{c\zeta(a_0 + a_1\zeta^{-1} + a_2\zeta^{-2})}{1 + n\zeta^{-2}}, f(\zeta) = \frac{c^2(b_0 + b_1\zeta^{-1} + b_2\zeta^{-2})}{1 + n\zeta^{-2}} \quad ...(35)$$

where a_0, b_0 are real constants and a_1, a_2, b_1, b_2 are complex constants to be determined from the boundary conditions (15), (16) and (17) in the same manner as before. After considerable algebraic manipulation it is found that

$$qw = \left| \frac{z - z_0}{c} \right|^2 \log \frac{\rho_0 R_0'}{R_0} + 2(1 - 3n^2 + (1 + n^2) m^2) J_1 \log \rho_0 \log \rho$$
$$+ \frac{J_1 g(\zeta_0) \log \rho}{| 1 + n\zeta_0^{-2} |^2} + \frac{J_1 g(\zeta) \log \rho_0}{| 1 + n\zeta^{-2} |^2} + \frac{p_1(1 - \rho_0^{-2})(1 - \rho^{-2})}{\rho_0^2 \rho^2 | 1 + n\zeta_0^{-2} |^2 | 1 + n\zeta^{-2} |^2}$$
$$\times Re\, [nJ_1 \{np_1 p_2 (\rho^2 + \rho_0^2) - p_1^2 \rho_0^2 \rho^2 - n^2 p_2^2$$
$$+ (1 - n^2) ((p_1 \rho_0^2 - np_2) \zeta^2 + (p_1 \rho^2 - np_2) \zeta_0^2) + np_1(1 + n^2) \zeta_0^2 \zeta^2$$
$$- (1 + mn - 3n^2 + mn^3) \bar{\zeta}_0^2 \zeta^2\} + J_2 \{((1 - 3mn + m^2 + nm^3) n^2$$
$$+ n(1 + mn) p_1 (\rho^2 + \rho_0^2) + (1 + mn - 3n^2 + mn^3) \rho_0^2 \rho^2) \zeta_0 \zeta$$
$$+ n(2np_1 (1 + \rho_0^2 \rho^2) + (1 - 2n^2 + m^2 n^2)(\rho^2 + \rho_0^2)) \bar{\zeta}_0 \zeta\}] \qquad \text{...(36)}$$

where

$$g(\zeta) = (1 - \rho^{-2}) \{m(2n - m - 6n^3 + 4mn^2 + mn^4) - \rho^2 J_1^{-1}$$
$$- n^2(1 - n^2 - 4mn + 3m^2 + m^2 n^2)(1 + \rho^{-2})$$
$$- n(1 - 3n^2 + m^2 + m^2 n^2) \rho^{-2} (\zeta^2 + \bar{\zeta}^2)\} \qquad \text{...(37)}$$

$$J_1^{-1} = 1 - 4n^2 - n^4 + 2mn(1 + n^2), \; J_2^{-1} = (1 + mn)^2 - 4n^2,$$
$$p_1 = m - n, \; p_2 = 1 - mn. \qquad \text{...(38)}$$

For $m = 0$ the expression (36) agrees with that given by eqn. (4.30) of Bassali (1960b) and for $n = 0$ it reduces to (32). For $m = n$ the hole is bounded by $| z | = c$ and (36) simplifies to (29).

References

Bassali, W. A. (1960a). Some problems in the small deflexions of clamped thin isotropic plates. *Z. angew. Math. Mech.*, **40**, 493-507.

——— (1960b). Green's functions for thin perforated elastic slabs. *Mathematika*, **7**, 101-12.

Bassali, W. A., and Dawoud, R. H. (1957). Green's functions for thin isotropic plates containing holes. *Proc. Camb. phil. Soc.*, **53**, 755-63.

Dean, W. R. (1953). The Green's function of an elastic plate. *Proc. Camp. phil. Soc.*, **49**, 319-26.

——— (1954). Green's function of an elastic plate. *Proc. Camb. phil. Soc.*, **50**, 623-27.

Dean, W. R., and Harris, G. Z. (1954). Green's function of an elastic plate. *Mathematika*, **1**, 18-23.

Elsirafy, I. H. M. (1968). Some problems in the torsion of elastic beams and the bending of normally loaded thin elastic plates. M.Sc. Thesis, Faculty of Science, Alexandria Univ., Appendix II, pp. 144-54.

Symonds, P. S. (1946). Concentrated force problems in plane strain, plane stress and transverse, bending of plates. *J. appl. Mech.*, **13**, 183-98.

Timoshenko, S., and Woinowsky-Krieger, S.(1959). Theory of Plates and Shells, 2nd ed. New York.