UNSTEADY LAMINAR CONVECTION IN UNIFORMLY HEATED VERTICAL PIPES II

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The problem of Gupta (1973) has been considered, in case source of heat generation is linearly dependent upon temperature. A new non-dimensional parameter $\delta = \beta a^2/k$ (β being the temperature coefficient of resistance) is introduced into the analysis. In case (when upflow is cooled or downflow is heated) the critical value of the Rayleigh number is considerably decreased if $\delta > 0$, of course for $\delta < q_1^2$ (square of the first zero of Bessel function or the Radial Mathieu function). In case of the circular pipe it is decreased from 33 to 9 while in case of elliptical pipe it is decreased from 1872 to 162. In the case of coaxial pipe R_c is decreased but not much. It is decreased from 60762 to 60515 when $c = a/b = 1.2$. But if $\delta < 0$ ($-ve$) the value of the critical Rayleigh number can be increased considerably for the same negative value of δ as for positive value. It has been found that the value of R_c is increased from 9 to 57, when it is a circular pipe while that for elliptical pipe it is 3592 instead of 162. The results available in the literature (Gupta 1973, Morton 1960) are the particular cases of these generalized results. During the course of this investigation we have found general solutions for all the three ducts : (1) circular, (2) elliptical and (3) coaxial pipes taking into consideration the source of heat generation dependent upon space coordinates and time arbitrarily, while the pressure gradient to be the arbitrary function of time only. When both the heat source and pressure gradient are absolute constants, the problem has been discussed in detail both for circular and elliptical pipes.

A new infinite series with the help of the transform involving Mathieu functions developed by Gupta (1964) has been summed up; another new infinite series in the case of circular pipe has also been summed up. Various non-dimensional quantities (for both the ducts) such as Nusselt number, volume rate of heat transfer, mean temperature have been calculated when both pressure gradient and source of heat generation are also absolute constants.

1. INTRODUCTION

Heat transfer problems of forced and free convection have attracted the attention of several research workers, because of their practical utility to physical problems. In these problems thermal convection consists of the transport of heat by a moving fluid in which the vertical variations of the temperature and hence density produce a distributed buoyancy force that itself modifies the flow. This interaction of velocity and temperature is an essential feature of convection, hence to find the convection flow in a heated flow, both the temperature and the velocity fields must be determined throughout whole region of the flow. Generally the research workers have taken source of heat generation to be either zero or absolute constant, but here we have taken source of heat generation to be linearly dependent upon temperature.

The energy equation and momentum equation are not uncoupled as in the case of constant wall temperatures. Hence this problem is not so simple mathematically. Though Tao (1961) suggested a method of complex analysis to solve the problem for steady state but in the case of unsteady case the equations are very much involved.

The main aim of this equation is to solve the unsteady case for various cross-sections with the help of the transform calculus. Both the finite Hankel transform and transform involving Mathieu functions have been used freely. The new infinite series have been summed up with the help of these transforms.

The striking feature of these investigations is that in case $\delta > 0$ the critical value of Rayleigh number is decreased to 162 on the increase of ellipticity e while if $\delta < 0$, the critical value of Rayleigh number increases to 3592 for the same numerical value of δ (i.e. positive and negative). The similar type of phenomenon occurs in each of the circular and coaxial pipes, when $\delta > 0$ i.e. positive, the steady state is reached very slowly. It can be observed from the Figs. 1-11 that the steady state is reached soon in the case of $\delta < 0$ in contrast to $\delta > 0$ for the same values of Rayleigh number.

2. Formulation of the Problem

The problem of Gupta (1973) has been considered here taking into account the effects of the source or sink, being dependent upon temperature. Here we have taken the source of heat generation to be linearly dependent upon the temperature and upon time as well as space coordinates.

The equations of continuity, momentum and energy are written using the Boussinesq approximation, i.e.

$$\partial_j u_j = 0$$

$$\partial_t u_i + u_j \partial_j u_i = \frac{1}{\rho_0} \partial_i p - p/\rho_0 \, g \vec{\lambda}_i + \nu \nabla^2 u_i$$

$$\partial_t T + u_j \partial_j T = k \nabla^2 T + Q$$

where we have used the summation convention and the following notation

$$\partial_i = \frac{\partial}{\partial x_i}\,;\ \partial_t = \frac{\partial}{\partial t}\ (j = 1, 2, 3;\ i = 1, 2, 3)$$

$\vec{\lambda}_i$ being the unit vector in the direction opposite to that of gravity; Q is the heat source. All other symbols have their usual meaning. Let the inner wall of the cylinder be held at temperature T_1, and T_0 is the temperature of the pipe at the level of the origin. Let a be the characteristic length, and let us suppose the temperature to be of the form

$$T - T_0 = -\beta' x_j \vec{\lambda}_j + \theta$$

where the first term, in which $\beta' = (T_0 - T_1)/a$ describes the temperature in the static state and θ is the deviation from the linear distribution. Also let Q the source of heat generation be taken as

$$Q = A_0(x, y, t) + \beta\theta$$

where A_0 is any arbitrary function and β is a constant. The fundamental equation of state is, which we approximate as,

$$\rho = \rho_0 [1 - \alpha(T - T_0)].$$

In this way we get the following system of equations

$$\partial_i u_i = 0 \quad \ldots(2.1)$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \tilde{\omega} + \alpha g \theta \vec{\lambda}_i + \nu \nabla^2 u_i \quad \ldots(2.2)$$

$$\partial_t \theta + u_j \partial_j \theta = \beta' \lambda_j u_i + k \nabla^2 \theta + \beta\theta + A_0(x, y, t) \quad \ldots(2.3)$$

where

$$\tilde{\omega} = p/\rho_0 + g x_j \vec{\lambda}_j - \tfrac{1}{2} \beta' \alpha g x'_k \vec{\lambda}_k x_j \vec{\lambda}_j$$

and α, g, ν, k are assumed to be constants.

To obtain the non-dimensional form of the equation we get

$$u_i = \frac{u'_i k}{a^2};\ \theta = \frac{k\theta'}{\alpha g a^3},\ t = \frac{a^2 t'}{k},\ x_i = a x'_i,\ \tilde{\omega} = k^2 \frac{\tilde{\omega}'}{a^2}.$$

This yields after dropping the primes

$$\partial_j u_j = 0$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \tilde{w} + P\lambda_i \theta + P\nabla^2 u_i$$

$$\partial_t \theta + u_j \partial_j \theta = R u_i + \nabla^2 \theta + \delta\theta + \frac{\alpha g a^5}{k^2} A_0(x, y, t)$$

where $P = \frac{\nu}{k}$ is the Prandtl number, $R = \frac{\alpha g \beta' a^4}{k}$ is the Rayleigh number and $\delta = \frac{\beta a^2}{k}$, $M = \alpha g a^5/k^2$.

Suppose the fluid is flowing in the direction of the axis of the cylinder and say, u_3 be the velocity in the direction of the axis of the pipe, then $u_1 = 0 = u_2$ and the equations are transformed into

$$\left.\begin{aligned} \frac{\partial u_3}{\partial x_3} &= 0 \\ \frac{\partial u_3}{\partial t} &= -\frac{\partial \tilde{\omega}}{\partial x_3} + P\nabla_1^2 u_3 + P\theta \end{aligned}\right\} \qquad \ldots(2.4)$$

$$\frac{\partial \theta}{\partial t} = Ru_3 + (\nabla_1^2 + \delta)\,\theta + Mf(x_1, x_2)\,\psi(t)$$

where

$$A_0\,(x_1, x_2, t) = \psi(t)\cdot f(x_1, x_2),\ \nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

In this one dimensional problem, where the conditions vary in time but not in the x_3-direction (apart from the pressure gradient), the equations become linear. It is quite evident that differentiation of second equation of (2.4) with respect to x_3 gives grad $\left(\frac{\partial}{\partial x_3}\right)$ to be zero, hence we conclude that $\frac{\partial \tilde{\omega}}{\partial x_3}$ is a function of time only. Therefore (2.4) can be written in the following form

$$\frac{\partial u_3}{\partial t} = P\nabla_1^2\, u_3 + P\theta - \phi(t) \qquad \ldots(2.5)$$

$$\frac{\partial \theta}{\partial t} = Ru_3 + \nabla_1^2\,\theta + \delta\theta + Mf(x_1, x_2)\,\psi(t). \qquad \ldots(2.6)$$

These equations are to be solved under the following boundary conditions

(i) $u_3 = 0 = \theta$ on the rigid boundary

(ii) $u_3 = 0 = \theta$ initially.

3. Circular Tubes

We have cosidered the case of circular pipe in this section. Since all quantities are independent of ϕ, and depend on r owing to axial symmetry. Here

$$u_3 = 2\sum_i \left[\frac{F'(0) - \beta F(0)}{\beta - \alpha}\, e^{\alpha t} + \frac{\alpha F(0) - F'(0)}{\beta - \alpha}\, e^{\beta t} + F(t)\right] \frac{J_0(rq_i)}{[J_1(q_i)]^2} \qquad \ldots(3.1)$$

where

$$F(t) = e^{\alpha t} \int^t e^{(\beta-\alpha)t} \int^t \chi(t)\, e^{-\beta t}\, (dt)^2$$

$$\chi(t) = P\left[M\bar{g}(q_i)\,\psi(t) - \phi'(t)\,\frac{J_1(q_i)}{q_i} - \left[\left(q_i^2 - \delta\right) q_i\right] \Big/ \phi(t)\, J_1(q_i)\right]$$

and

$$\alpha = \tfrac{1}{2}\left\{\delta - (P+1)\, q_i^2 + \left[\left\{\delta + (P-1)\, q_i^2\right\}^2 + 4RP\right]^{1/2}\right\}$$

$$\beta = \tfrac{1}{2}\left\{\delta - (P+1)\, q_i^2 - \left[\left\{\delta + (P-1)\, q_i^2\right\}^2 + 4RP\right]^{1/2}\right\} \qquad \ldots(3.2)$$

and q_i is the ith positive root of $J_0(q) = 0$ ...(3.3)

and

$$\theta = 2 \sum_i \left[\frac{G'(0) - \beta G(0)}{\beta - \alpha}\, e^{\alpha t} - \frac{G'(0) - \alpha G(0)}{\beta - \alpha}\, e^{\beta t} + G(t)\right] \frac{J_0(rq_i)}{(J_1(q_i))^2} \qquad \ldots(3.4)$$

where

$$G(t) = e^{\alpha t} \int^t e^{(\beta-\alpha)t} \int e^{-\beta t}\, Y(t)\, (dt)^2$$

and

$$Y(t) = MP\, q_i^2\, \bar{g}(q_i)\, \psi(t) + M\psi'(t)\, \bar{g}(q_i) - R\phi(t)\, \frac{J_1(q_i)}{q_i}. \qquad \ldots(3.5)$$

The summation in (3.1) and (3.4) being extended over all the positive roots of (3.3). Both for velocity fields and temperature fields, it can readily be inferred that as R (the Rayleigh number) increases the velocity as well as temperature stabilizes but as soon as R reaches the critical value i.e. $R_c = q_i^4 - \delta q_i^2$ [q_i being the first zero of (3.3)] the steady state is disturbed.

It shows that

$$R < R_c = (2.403)^4 - \delta(2.403)^2$$

i.e. we infer that the critical value of the Rayleigh number is decreased if $\delta > 0$ (i.e. as in most cases heating by electric current). The Rayleigh number is increased if $\delta < 0$ (e.g. in heating by electric current in a material with a negative temperature coefficient of resistance, or in the case of removal by radiation or convection). See table given below .

Source					Sink				
δ	1	2	3	4	$-\delta$	1	2	3	4
R_c	27	21	15	9	R_c	39	45	51	57

4. Particular Cases

Let us suppose that

$$f(t) = EPe^{-\gamma t},\quad \psi(t) = Fe^{-st},\quad \bar{g}(q_i) = \frac{J_1(q_i)}{q_i}. \qquad \ldots(4.1)$$

Thus we find that

$$u_3 = \frac{Ek_1 - M\hat{F}}{(k_1 + \delta)(\delta^2 + 4R)^{1/2}} \left\{1 - \frac{J_0(r\sqrt{(\delta + k_1)})}{J_0((k_1 + \delta)^{1/2})}\right\}$$

$$- \frac{Ek_2 - M\hat{F}}{(k_2 + \delta)(\delta^2 + 4R)^{1/2}} \left\{1 - \frac{I_0(r\sqrt{-(\delta + k_2)})}{I_0(\sqrt{-(\delta + k_2)})}\right\}$$

$$+ 2 \sum_i \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha} \times \frac{M\hat{F} + E\left(\delta - q_i^2\right)}{q_i^2\left(q_i^2 - \delta\right) - R} \times \frac{J_0(rq_i)}{q_i J_1(q_i)} \qquad \ldots(4.2)$$

and

$$\theta = - \frac{k_2(Ek_1 - M\hat{F})}{(k_1 + \delta)(\delta^2 + 4R)^{1/2}} \left[1 - \frac{J_0(r\sqrt{(\delta + k_1)})}{J_0(\sqrt{(\delta + k_1)})}\right]$$

$$+ \frac{k_1(Ek_2 - M\hat{F})}{(k_2 + \delta)\sqrt{(\delta^2 + 4R)}} \times \left[1 - \frac{I_0(r\sqrt{-(\delta + k_2)})}{I_0(\sqrt{-(\delta + k_2)})}\right]$$

$$+ 2 \sum_i \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha} \frac{M\hat{F}q_i^2 - RE}{\left(q_i^2 - \delta\right) q_i^2 - R} \frac{J_0(rq_i)}{q_i J_i(q_i)} \qquad \ldots(4.3)$$

where

$$k_1 = \frac{-\delta + \sqrt{(\delta^2 + 4R)}}{2}, \; k_2 = \frac{-\delta - \sqrt{(\delta^2 + 4R)}}{2}.$$

As was conjectured by Morton (1960) when $\delta = 0 = F$ the velocity as well as temperature become steady ultimately provided that the Rayleigh number is less than R_c the critical value. In case $\delta = 0 = F$ our results are in complete agreement with Morton and if $\delta = 0$ to that of Gupta (1973). It can be readily shown that if the temperature gradient increases with height both the fields can be obtained by changing R into $-R$, and in this case the critical value of R is high and the flow becomes ultimately steady. Here our θ is $1/R$ that of Morton (1960). In case $\delta = q_i^2 > 0$ then $R = 0$ is the critical value of R_c i.e. $\delta < q_i^2$.

5. Coaxial Pipes

Proceeding exactly in the same way as in §4 we have

$$\bar{u}_3 = \frac{F_1'(0) - \beta F_1(0)}{\beta - \alpha} e^{\alpha t} + \frac{F_1(0) - \alpha F_1(0)}{\beta - \alpha} e^{\beta t} + F_1(t) \qquad \ldots(5.1)$$

where

$$F_1(t) = e^{\alpha t} \int^{t} e^{(\beta-\alpha)t} \int \chi_1(t)\, e^{-\beta t}\, (dt)^2$$

and

$$\chi_1(t) = PMg_1(q_i)\,\psi(t) - \left[\left(q_i^2 - \delta\right)\phi(t) + \phi'(t)\right] \times \frac{2}{\pi q_i^2}\left[\frac{J_0(cq_i)}{J_0(q_i)} - 1\right]$$

where $c = b/a;\ a < b;$

$$B_0(rq_i) = J_0(rq_i)\, Y_0(q_i) - Y_0(rq_i)\, J_0(q_i) \qquad \text{...(5.2)}$$

while q_i is the ith root of the equation

$$J_0(cq_i)\, Y_0(q_i) - Y_0(cq_i)\, J_0(q_i) = 0 \qquad \text{...(5.3)}$$

α and β have the same meaning as before.

Equation (5.1) is inverted to give

$$u_3 = \frac{\pi^2}{2} \sum_i \frac{q_i^2\, J_0^2\,(cq_i)}{J_0^2\,(q_i) - J_0^2\,(cq_i)} \left[\frac{F_1'(0) - \beta F_1(0)}{\beta - \alpha}\, e^{\alpha t} - \frac{F_1'\,(0) - \alpha F(0)}{\beta - \alpha}\, e^{\beta t} + F_1(t)\right] B_0(rq_i). \qquad \text{...(5.4)}$$

Since we are interested in the case of constant pressure gradient and constant heat source strength, we write

$$\phi_1(t) = \frac{2P}{\pi q_i^2}\left(\frac{J_0(cq_i)}{J_0(q_i)} - 1\right)\left(M\hat{F} - E\left(q_i^2 - \delta\right)\right)$$

$$F_1(t) = \frac{2\left[M\hat{F} - E\left(q_i^2 - \delta\right)\right]}{\pi q_i^2\left[\left(q_i^2 - \delta\right) q_i^2 - R\right]}\left[\frac{J_0(q_i c)}{J_0(q_i)} - 1\right] = F_1(0)$$

It is quite obvious that $F_1'\,(0) = \phi(0) = \psi(0) = 0$. Thus we get

$$u_3 = \pi \sum_i \left(1 - \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha}\right) \frac{M\hat{F} - E\left(q_i^2 - \delta\right)}{\left(q_i^2 - \delta\right) q_i^2 - R} \times \frac{J_0(q_i c)}{J_0(cq_i) + J_0(q_i)}\, B_0(rq_i) \qquad \text{...(5.5)}$$

and similarly

$$\theta = \pi \sum_i \left(1 - \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha}\right) \frac{M\hat{F} - RE}{\left(q_i^2 - \delta\right) q_i^2 - R}$$

$$\times \frac{J_0(q_i c)}{J_0(cq_i) + J_0(q_i)} B_0(rq_i) \qquad \ldots(5.6)$$

where summation is being extended over the positive roots of (5.3). It is quite obvious that both velocity and temperature fields become unsteady and infinite at the smallest root of (5.3) i.e. $R = q_i^4 - \delta q_i^2$ and beyond this value of R, which is the critical value of R_c, both the fields become non-laminar. The value of R_c for different values of c and δ are given in the table. It is quite evident that Rayleigh number increases for negative δ and decreases for positive δ (i.e. if it is a sink R_c increases and decreases if it is a source). The same type of behaviour as in Gupta (1973) continues i.e. if the gap between the cylinder decreases R increases and if gap increases R is decreased. Hence we conclude that the value of critical Rayleigh number can be made as large as possible on the increase of the strength of the sink and may be further decreased on the increase of the source.

If $\delta > 0$, $q_i^4 - \delta q_i^2 = R_c$

c	1.2	1.5	2.0	2.5	3.0	3.5	4.0
q_i	15.7014	6.2702	3.1230	2.0372	1.5485	1.2339	1.0244
R_c	60513	1505	85	14	3.37	0.8	0.04

If $\delta < 0$, $q_i^4 + q_i^2 \delta = R_c$

c	1.2	1.5	2.0	2.5	3.0	3.5	4.0	
q_i	15.7014	6.2702	3.1230	2.0732	1.5485	1.2339	1.0244	
R_c	61007	1585	105	22	8.15	3.81	2.04	$\delta = 1$
R_c	61254	1625	115	26	10.52	5.31	3.08	$\delta = 2$
R_c	61501	1665	125	30	12.89	6.81	4.12	$\delta = 3$

6. Unsteady Flow within a Hollow Elliptical Cylinder

In this section the same problem has been discussed for the case of the elliptical cylinder. Here

$$u_3 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[\frac{F_2'(0) - \beta F_2(0)}{\beta - \alpha} e^{\alpha t} + \frac{\alpha F_2(0) - F_2'(0)}{\beta - \alpha} e^{\beta t} + F_2(t) \right]$$

$$\times \frac{Se_{2n}(\eta, q_{2n,m}) \, Je_{2n}(\xi, q_{2n,m})}{\pi \int_0^{\xi_0} J_{2n}^2(\xi, q_{2n,m}) \, [Me_{2n}(q_{2n,m}) \cosh 2\xi - \theta_{2n,m}] \, d\xi} \quad \ldots(6.1)$$

where

$$\Theta_{2n,m} = \int_0^{2\pi} Se_{2n}^2(\eta, q_{2n,m}) \cos 2\eta \, d\eta = \pi \left[D_0^{2n} D_2^{2n} + \sum_{r=0}^{\infty} D_{2r}^{(2n)} D_{2r+2}^{2n} \right]$$

$$Me_{2n}(q_{2n,m}) = \int_0^{2\pi} Se_{2n}^2(\eta, q_{2n,m}) \, d\eta$$

where

$$F_2(t) = e^{\alpha t} \int^t e^{(\beta - \alpha)} \chi(t) \int^t e^{\alpha t} (dt)^2$$

$$\chi(t) = P\{M\psi(t) \, \bar{g}(q_{2n,m}) - [\bar{\phi}'(t, q_{2n,m}) + (\gamma_{2n,m}^2 - \delta) \, \bar{\phi}(t, q_{2n,m})]\} \text{ (say)}$$

and

$$\alpha = \tfrac{1}{2} \{\delta - (P + 1) \gamma_{2n,m}^2 + \sqrt{[\{\delta + (P - 1) \gamma_{2n,m}^2\}^2 + 4RP]}\}$$

$$\beta = \tfrac{1}{2} [\delta - (P + 1) \gamma_{2n,m}^2 - \sqrt{\{(\delta + (P - 1) \gamma_{2n,m}^2)^2 + 4RP\}}]$$

$q_{2n,m}$ is the mth positive zero of $Je_{2n}(\xi_0, q) = 0$. ...(6.2)

Similarly for θ we get

$$\theta = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left[\frac{G_2'(0) - \beta G_2(0)}{\beta - \alpha} e^{\alpha t} - \frac{G_2'(0) - \alpha G_2(0)}{\beta - \alpha} e^{\beta t} + G_2(t) \right]$$

$$\times \frac{Se_{2n}(\eta, q_{2n,m}) \, Je_{2n}(\xi q_{2n,m})}{\int_0^{\xi_0} Je_{2n}^2(\xi, q_{2n,m}) \, [Me_{2n} \cosh 2\xi - \Theta_{2n,m}] \, d\xi} \quad \ldots(6.3)$$

where

$$Y_1(t) = MP\gamma_{2n,m}^2 \psi(t) \, \bar{g}(q_{2n,m}) - M\bar{\psi}'(t) \, \bar{g}(q_{2n,m}) - R\phi(t, q_{2n,m})$$

$$G_2(t) = e^{\alpha t} \int^t e^{(\beta - \alpha)t} \int^t e^{\beta t} Y_1(t) \, (dt)^2.$$

7. Transition to Circular Cylinder

We see that both u_3 and θ are independent of ϕ when we consider the case of the circular cylinder of radius unity. We have $\gamma^2_{2n,m} = q_{0,m}$, $m = 1, 2 \ldots$ these being the zeros of the roots of $J_0(q) = 0$. Also when $e = 0$ and $\xi \to \infty$, $\sinh \xi \to \cosh \xi$ $\times$ $h \cosh \xi \to r$, $h \sinh \xi \, d\xi \to dr$, $\cosh 2\xi \, d\xi \to 2r/h^2 \, dr$.
Making use of these we find

$$Se(\eta, q_{2n,m}) \to 1,\; Je_{2n}(\xi, q_{2n,m}) \to J_0(rq).$$

Hence (6.1) and (6.3) degenerate to (3.1) and (3.4) respectively as the elliptical cylinder degenerates into a circular cylinder.

8. A Particular Case

If the strength of the source as well as the pressure gradient are absolute constants then

$$u_3 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{MF + E(\delta - \gamma^2_{2n,m})}{(\gamma^2_{2n,m} - \delta)\,\gamma^2_{2n,m} - R} \left[1 - \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta}\right] \times \frac{Se_{2n}(\eta, q_{2n,m})\, Je_{2n}(\xi, q_{2n,m})}{\int_0^{\xi_0} J^2_{2n}(\xi, q_{2n,m})\,[Me_{2n} \cosh 2\xi - \Theta_{2n,m}]\, d\xi} \qquad \ldots(8.1)$$

and

$$\theta = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{M\hat{F}\gamma^2_{2n,m} - RE}{(\gamma^2_{2n,m} - \delta)\,\gamma^2_{2n,m} - RE} \left(1 - \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\alpha - \beta}\right) \times \frac{Se_{2n}(\eta, q_{2n,m})\, Je_{2n}(\xi, q_{2n,m})}{\int_0^{\xi_0} Je^2_{2n}(\xi, q_{2n,m})\,[Me_{2n}(q_{2n,m}) \cosh 2\xi - \Theta_{2n,m}]\, d\xi} \qquad \ldots(8.2)$$

where

$$\gamma^2_{2n,m} = (q_{2n,m}/h^2).$$

Now we sum the series for the steady state terms with the help of the results obtained in the Appendix A from (8.1), by virtue of the boundary conditions we have (see appendix A)

$$u_3 = \frac{2(Ek_1 - \hat{F})}{(\sqrt{(\delta^2 + 4R)} + \delta)\,(\delta^2 + 4R)^{1/2}} \left[1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q_1)\, Je_{2n}(\xi, q_1)\, Se_{2n}(\eta, q_1)}{Je_{2n}(\xi_0, q_i)\, Me_{2n}(q_1)}\right]$$

(equation continued on p. 1355)

$$+\frac{2(Ek_2 - M\hat{F})}{\sqrt{\delta^2 + 4R}\,(\sqrt{(\delta^2 + 4R)} - \delta)}\left[1 - \sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(-q_2)\,Je_{2n}(\xi, -q_2)\,Se_{2n}(\eta, q_2)}{Je_{2n}(\xi_0, -q_1)\,Me_{2n}(-q_2)}\right]$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta e^{\alpha t} - \alpha e^{\beta t}}{\alpha - \beta}$$

$$\times\frac{[MF + (\delta - \gamma_{2n,m}^2)]\,Je_{2n}(\xi, q_{2n,m})\,Se_{2n}(\eta, q_{2n,m})}{[(\gamma_{2n,m}^2 - \delta)\,\gamma_{2n,m}^2 - R]\int_0^{\xi_0} Je_{2n}^2(\xi, q_{2n,m})\,\{Me_{2n}(q_{2n,m})\cosh 2\xi - \Theta_{2n,m}\}\,d\xi}$$

...(8.3)

where

$$q_1 = h^2\left(\frac{\delta + \sqrt{(\delta^2 + 4R)}}{2}\right),\quad q_2 = \left(\frac{\sqrt{\delta^2 + 4R)} - \delta}{2}\right)h^2$$

and similarly

$$\theta = -\frac{2k_2(Ek_1 - M\hat{F})}{(\delta + \sqrt{(\delta^2 + 4R)})\sqrt{(\delta^2 + 4R)}}$$

$$\times\left[1 - \sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(q_1)\,Je_{2n}(\xi, q_1)\,Se_{2n}(\eta, q_1)}{Je_{2n}(\xi_0, q_1)\,Me_{2n}(q_1)}\right]$$

$$-\frac{2k_1(Ek_2 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} - \delta)(\delta^2 + 4R)^{1/2}}$$

$$\times\left[1 - \sum_{n=0}^{\infty}\frac{2\pi D_0^{2n}(-q_2)\,Je_{2n}(\xi, -q_2)\,Se_{2n}(\eta, -q_2)}{Je_{2n}(\xi_0, -q_2)\,Me_{2n}(-q_2)}\right]$$

$$-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta e^{\alpha t} - \alpha e^{\beta t}}{\beta - \alpha}$$

$$\times\frac{M\hat{F}\gamma_{2n,m}^2 - R}{(\gamma_{2n,m}^2 - \delta)\,\gamma_{2n,m}^2 - R}\cdot\frac{Je_{2n}(\xi, q_{2n,m})\,Se_{2n}(\eta, q_{2n,m})}{\int_0^{\xi_0} Je_{2n}^2(\xi, q_{2n,m})\,[Me_{2n}(q_{2n,m})\cosh 2\xi - \Theta_{2n,m}]\,d\xi}$$

...(8.4)

It can be readily seen that as $\delta \to 0$ the results investigated above agree with those of Gupta (1973). It is quite obvious that in this case when R reaches its critical value, i.e. $R_c = (\gamma^2_{2n,m} - \delta)\,\gamma^2_{2n,m}$ both the temperature and velocity fields become unsteady. By changing R into $-R$ we obtain the results when the temperature gradient is in opposite direction.

$$R_c = \gamma^4_{2n,m}$$

$\delta = 0$

R_c	39	57.5	99.72	156.97	348.8	787.5	1872
ξ_0	1.573	1.044	0.748	0.603	0.457	0.346	0.268
e	$(2.5)^{-1}$	$(1.59)^{-1}$	$(1.29)^{-1}$	$(1.18)^{-1}$	$(1.10)^{-1}$	$(1.06)^{-1}$	$(1.04)^{-1}$
$q_{0,1}$	1	3	6	9	15	25	40

$$R_c = (\gamma^2_{2n,m} - \delta)\,\gamma^2_{2n,m}$$

$\delta > 0$

R_c	8	14	19	31	78	87	162
ξ_0	1.573	1.044	0.748	0.603	0.457	0.346	0.268
e	$(2.5)^{-1}$	$(1.59)^{-1}$	$(1.29)^{-1}$	$(1.18)^{-1}$	$(1.10)^{-1}$	$(1.06)^{-1}$	$(1.04)^{-1}$
$q_{0,1}$	1	3	6	9	15	25	40
δ	5	6	8	10	15	25	40

$$R_c = (\gamma^2_{2n,m} - \delta)\,\gamma^2_{2n,m}$$

$\delta < 0$

R_c	60	100	176	282	518	1487	3592
ξ_0	1.573	1.044	0.748	0.603	0.457	0.346	0.268
e	$(2.5)^{-1}$	$(1.59)^{-1}$	$(1.29)^{-1}$	$(1.18)^{-1}$	$(1.10)^{-1}$	$(1.06)^{-1}$	$(1.04)^{-1}$
$q_{0,1}$	1	3	6	9	15	25	40
δ	-5	-6	-8	-10	-15	-25	-40

9. General Discussion of the Solution

In all the cases both for velocity as well as temperature fields, the solution consists of two parts. One is the transient part while the other is the steady part. It is quite obvious that the transient part decreases as time increases (provided that $R < R_c$ the critical value, when the down flow is heated the solution for u_3 and θ both become unsteady and non-laminar on the increase of the Rayleigh number. It has been seen that the Rayleigh number for different boundaries is different and depends upon the parameter δ and its sign as well. The striking feature of the solution obtained here is that in case $\delta < 0$ Rayleigh number can be increased considerably on the increase of δ (i.e. the strength of the sink is sufficiently great). In case $\delta > 0$ the Rayleigh number is decreased considerably on the increase of the strength of the source which can be easily seen from the tables. Thus as in Gupta (1973) we conclude that the fully developed flow which is found for small Rayleigh numbers becomes impossible as the Rayleigh number approaches the critical value.

Circular Tube

Following the method of Gupta (1973), if $P = 1$, then

$$u_3 \cong \Bigg[\frac{2(Ek_1 - M\hat{F})}{(\delta^2 + 4R)^{1/2}\,(\delta + \sqrt{(\delta^2 + 4R)}}\left\{1 - \frac{J_0(r\sqrt{(k_1 + \delta)}}{J_0(\sqrt{(k_1 + \delta)})}\right\} + \frac{2(Ek_2 - M\hat{F})}{\sqrt{(\delta^2 + 4R)}\,[\sqrt{(\delta^2 + 4R)} - \delta]}\left\{1 - \frac{I_0[r\sqrt{-(\delta + k_2)}]}{I_0(\sqrt{-(k_2 + \delta)})}\right\}\Bigg] \times \Bigg[\Bigg(1 - \tfrac{1}{2}\Bigg[\left\{\frac{(\delta - 2q_1^2 + \sqrt{(\delta^2 + 4R)})}{\sqrt{(\delta^2 + 4R)}}\right\}\exp\left\{\frac{\delta - 2q_1^2 - \sqrt{(\delta^2 + 4R)}}{2}\right\}t - \frac{(\delta - 2q_1^2 - \sqrt{(\delta^2 + 4R)}}{\sqrt{(\delta^2 + 4R)}}\exp\left\{\frac{\delta - 2q_1^2 + \sqrt{(\delta^2 + 4R)}}{2}\right\}t\Bigg)\Bigg] \quad ...(9.1)$$

following the same method as for u_3 we have for temperature difference

$$\theta \cong \Bigg[\frac{-2k_2(Ek_1 - M\hat{F})}{\{(\delta + \sqrt{(\delta^2 + 4R)}\}\,(\delta^2 + 4R)^{1/2}}\left(1 - \frac{J_0(r\sqrt{(k_1 + \delta)})}{J_0(\sqrt{(k_1 + \delta)})}\right) - \frac{2k_1(Ek_2 - M\hat{F})}{\sqrt{\delta^2 + 4R}\,\{\sqrt{(\delta^2 + 4R)} - \delta\}}\left(1 - \frac{I_0(r\sqrt{-(k_2 + \delta)})}{I_0(\sqrt{-(k_2 + \delta)})}\right)\Bigg] \times \Bigg[1 - \tfrac{1}{2}\left\{\frac{(\delta - 2q_1^2 + \sqrt{(\delta^2 + 4R)}}{\sqrt{(\delta^2 + 4R)}}\exp\left(\frac{\delta - 2q_1^2 - \sqrt{\delta^2 + 4R}}{2}\right)t - \frac{(\delta - 2q_1^2 - \sqrt{\delta^2 + 4R})}{\sqrt{\delta^2 + 4R}}\exp\left(\frac{\delta - 2q_1^2 + \sqrt{(\delta^2 + 4R)}}{2}\right)t\right\}\Bigg] \quad ...(9.2)$$

where q_1 is the first and smallest zero of $J_0(q) = 0$.

It is quite evident from (9.1) and (9.2) that as R increases the transient part dies out slowly and if it reaches its critical value and, or passes beyond this then both the velocity and temperature fields become unsteady and non-laminar.

Elliptic Pipe

Following the same method as in the case of circular pipe we have from (8.3) and (8.4)

$$u_3 \cong \left[\frac{2(Ek_1 - M\hat{F})}{\sqrt{\delta^2 + 4R}\,\{\sqrt{(\delta^2 + 4R)} + \delta\}} \right.$$

$$\times \left\{ 1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q_1)\, Je_{2n}(\xi, q_1)\, Se_{2n}(\eta, q_1)}{Je_{2n}(\xi_0, q_1)\, Me_{2n}(q_1)} \right\}$$

$$+ \frac{2(Ek_2 - M\hat{F})}{\sqrt{\delta^2 + 4R}\,\{(\delta^2 + 4R)^{1/2} - \delta\}}$$

$$\left. \times \left\{ 1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(-q_2)\, Je_{2n}(\xi_0, -q_2)\, Se_{2n}(\eta, -q_2)}{Me_{2n}(-q_2)\, Je_{2n}(\xi_0, -q_2)} \right\} \right]$$

$$\times \left[1 - \tfrac{1}{2} \left\{ \frac{\delta - 2\gamma_{2n,1}^2 + \sqrt{\delta^2 + 4R}}{\sqrt{\delta^2 + 4R}} \exp\left(\frac{\delta - 2\gamma_{2n,1}^2 - \sqrt{\delta^2 + 4R}}{2} \right) t \right. \right.$$

$$\left. \left. - \frac{\delta - 2\gamma_{2n,1}^2 - \sqrt{\delta^2 + 4R}}{\sqrt{(\delta^2 + 4R)}} \exp\left(\frac{\delta - 2\gamma_{2n,1}^2 + \sqrt{\delta^2 + 4R}}{2} \right) t \right\} \right]. \quad \ldots(9.3)$$

$$\theta \cong - \left[\frac{2k_2(Ek_1 - M\hat{F})}{(\delta^2 + 4R)^{1/2}(\delta + \sqrt{\delta^2 + 4R})} \right.$$

$$\times \left\{ 1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q_1)\, Je_{2n}(\xi, q_1)\, Se_{2n}(\eta, q_1)}{Je_{2n}(\xi_0, q_1)\, Me_{2n}(q_1)} \right\}$$

$$+ \frac{2k_1(Ek_2 - MF)}{(\sqrt{(\delta^2 + 4R)} - \delta)\,(\delta^2 + 4R)^{1/2}}$$

$$\left. \times \left\{ 1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(-q_2)\, Je_{2n}(\xi, -q_2)\, Se_{2n}(\eta, -q_2)}{Je_{2n}(\xi_0, -q_2)\, Me_{2n}(-q_2)} \right\} \right]$$

$$\times \left[1 - \tfrac{1}{2} \left\{ \frac{\delta - 2\gamma_{2n,1}^2 + \sqrt{\delta^2 + 4R}}{\sqrt{\delta^2 + 4R}} \exp\left(\frac{\delta - 2\gamma_{2n,1}^2 - \sqrt{\delta^2 + 4R}}{2} \right) t \right. \right.$$

$$\left. \left. - \frac{\delta - 2\gamma_{2n,1}^2 - \sqrt{\delta^2 + 4R}}{(\delta^2 + 4R)^{1/2}} \exp\left(\frac{\delta - 2\gamma_{2n,1}^2 + \sqrt{\delta^2 + 4R}}{2} \right) t \right\} \right]. \quad \ldots(9.4)$$

It is easily verified that as $\delta \to 0$ the results of Gupta (1973) are its particular cases. Besides this the transient part dies out slowly here also as in the case of the circular cylinder, and if it reaches its critical value or beyond that then the field (velocity and temperature) become non-laminar.

10. Convection when the Pipe Temperature Decreases with Height

In this section we shall discuss the case of both circular and elliptical tubes in case the pressure gradient and the source of heat generation are absolute constants. Various non-dimensional quantities have been calculated. A general formula for Nusselt number in the case of the unsteady flow has been investigated for both the tubes. We have discussed the case of $\delta > 0$ as well as of $\delta < 0$ (i.e. either source of heat generation or sink). When $\delta \to 0$ our results are in complete agreement with Gupta (1973) and also if $F \to 0$ with that of Morton. For unsteady case figures have been drawn both for velocity and temperature distributions.

The steady cooling of ascending hot fluid or the steady heating of descending cold fluid correspond, when R is positive.

Circular Tube

In case both pressure gradient and heat source are constant we have the following results.

(a) The rate of heat transfer through the pipe walls to the fluid per unit area of pipe surface is :

$$q = -K\left(\frac{\partial T}{\partial r}\right)_{r=1} = -K\frac{k}{\alpha g a^4}\left(\frac{\partial \theta}{\partial r}\right)_{r=1} = -\frac{K\beta'}{R}\left(\frac{\partial \theta}{\partial r}\right)_{r=1}$$

$$= -\frac{K\beta'}{R}\Bigg[\left\{\frac{+2k_2(Ek_1 - M\hat{F})}{\sqrt{\delta^2+4R}\,(\sqrt{\delta^2+4R}+\delta)}\left\{\frac{\sqrt{\delta^2+4R}+\delta}{2}\right\}^{1/2}\frac{J_0'(\sqrt{(k_1+\delta)})}{J_0(\sqrt{(k_1+\delta)})}\right.$$

$$\left. - \frac{2k_1(Ek_2 - M\hat{F})}{(\sqrt{(\delta^2+4R)}-\delta)\sqrt{(\delta^2+4R)}} \times \frac{I_0'(\sqrt{-(k_2+\delta)})\sqrt{-(k_2+\delta)}}{I_0(\sqrt{-(k_2+\delta)})}\right\}$$

$$-2\sum_i \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha}\,\frac{M\hat{F}q_i^2 - RE}{\left(q_i^2 - \delta\right)q_i^2 - R}\Bigg] \quad \ldots(10.1)$$

where the primes imply differentiation with respect to the argument and K is the thermal conductivity. In case $\delta \to 0$ these results agree with those of Gupta (1973), when $t \to \infty$ the first expression in curly brackets in (10.1) represents the steady state, which is quite a new result. If $\delta \to 0$, $F \to 0$ and $t \to \infty$ the above result agrees with that of Morton (1960).

(b) The rate of volume flow through the pipe is

$$\int_0^a 2\pi r u \, dr = 2\pi a k \int_0^1 r u \, dr$$

$$= (2\pi a k)\left[\left\{\frac{2(Ek_1 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} + \delta)\,(\delta^2 + 4R)^{1/2}}\right.\right.$$

$$\times \left\{\tfrac{1}{2} - \frac{\sqrt{2}}{((\sqrt{(\delta^2 + 4R)} + \delta)^{1/2}} \cdot \frac{J_1((k_1 + \delta)^{1/2})}{J_0((k_1 + \delta)^{1/2})}\right\}$$

$$+ \frac{2(Ek_2 - MF)}{\sqrt{(\delta^2 + 4R)}\,(\sqrt{(\delta^2 + 4R)} - \delta)}$$

$$\times \left\{\tfrac{1}{2} - \frac{\sqrt{2}}{(\sqrt{(\delta^2 + 4R)} - \delta)^{1/2}} \cdot \frac{I_1(\sqrt{-(k_2 + \delta)})}{I_0(\sqrt{-(k_2 + \delta)})}\right\}\Bigg\}$$

$$\left. + 2 \sum_i \frac{M\hat{F} + E\left(\delta - q_i^2\right)}{q_i^2\left(q_i^2 - \delta\right) - R} \cdot \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha} \cdot \frac{1}{q_i^2}\right]. \qquad \text{...(10.2)}$$

When $t \to \infty$ the flux in the steady state is given by first expression of (10.2) within curly brackets.

(c) A measure of effectiveness of heat transfer is provided by the Nusselt number.

$$N = \frac{\text{Rate of heat transfer per unit area of pipe wall} \times \text{Pipe dia.}}{K \times \text{Characteristic temperature in main direction of conduction}}. \qquad \text{...(10.3)}$$

This temperature can be measured in different ways; here, partly for comparison with the results quoted above, it will be taken as the difference between the wall temperature and the mean temperature, which is

$$[T]_{\text{mean}} = 1/\pi a^2 \int_0^a 2\pi\, r'\theta'\, dr' = \frac{2k}{\alpha g a^3} \int_0^1 r\theta\, dr = \frac{2\beta' a}{R} \int_0^1 r\theta\, dr$$

$$= \frac{2\beta' a}{R}\left[\left\{\frac{-2k_2(Ek_1 - M\dot{F})}{\sqrt{\delta^2 + 4R}\,\{\sqrt{(\delta^2 + 4R)} + \delta\}}\right.\right.$$

$$\left(\tfrac{1}{2} - \sqrt{\frac{2}{\sqrt{(\delta^2 + 4R)} + \delta}}\; \frac{J_1(\sqrt{k_1 + \delta})}{J_0(\sqrt{(k_1 + \delta)})}\right) -$$

(*equation continued on p.* 1361)

$$- \frac{2k_1(Ek_2 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} - \delta)\,(\delta^2 + 4R)^{1/2}}$$

$$\times \left(\frac{1}{2} - \sqrt{\frac{2}{\sqrt{(\delta^2 + 4R)} + \delta}}\; \frac{I_1(\sqrt{-(k_2 + \delta)})}{I_0(\sqrt{-(k_2 + \delta)})} \right) \Bigg\}$$

$$+ 2 \sum_i \frac{\alpha e^{\beta t} - \beta e^{\alpha t}}{\beta - \alpha} \cdot \frac{M\hat{F} q_i^2 - RE}{\left(q_i^2 - \delta\right) q_i^2 - R}\, \frac{1}{(q_i)^2} \Bigg] . \qquad \ldots(10.4)$$

Nusselt number is given by substituting for heat transfer and mean temperature in the formula (10.3) which is quite a new result.

We get N for steady state by taking $t \to \infty$, which is evidently a new result. As $\delta \to 0$ the results of Gupta (1973) are in complete agreement with our results.

Elliptical Tubes

Now we shall find all the above quantities for the case of elliptical tube.

(A) The rate of heat transfer through the pipe walls per unit area of a pipe surface is

$$\dot{q} = - \frac{K\beta}{RS} \int_0^{2\pi} \left(\frac{\partial \theta}{\partial \xi} \right)_{\xi = \xi_0} d\eta$$

S being the length of the perimeter of the ellipse. Hence

$$q = - \frac{K\beta'}{RS} \Bigg[\Bigg\{ \frac{2k_2(Ek_1 - M\hat{F})}{(\delta + \sqrt{(\delta^2 + 4R)})\,\sqrt{(\delta^2 + 4R)}}$$

$$\times \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n} (q_1)^2\, Je'_{2n}(\xi_0, q_1)}{Je_{2n}(\xi_0, q_1)\, Me_{2n}(q_1)}$$

$$+ \frac{2k_1 (Ek_2 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} - \delta)\,\sqrt{(\delta^2 + 4R)}}$$

$$\times \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n} (-q_2)^2\, Je'_{2n}(\xi_0, -q_2)}{Je_{2n}(\xi_0, -q_2)\, Me_{2n}(-q_2)} \Bigg\} +$$

(*equation continued on p.* 1362)

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta e^{\alpha t}-\alpha e^{\beta t}}{\beta-\alpha}$$

$$\times\frac{M\hat{F}\gamma^2_{2n,m}-RE}{((\gamma^2_{2n,m}-\delta)\,\gamma^2_{2n,m}-R)}\times 2\pi D_0^{2n}(q_{2n,m})\,Je_{2n}(\xi_0,q_{2n,m})$$

$$\left.\times\frac{1}{\int_0^{\xi_0} Je^2_{2n}(\xi,q_{2n,m})\,[\cosh 2\xi\, Me_{2n}(q_{2n,m})-\Theta_{2n,m}]\,d\xi}\right] \qquad \ldots(10.6)$$

where prime denotes differentiation with respect to ξ.

(B) The rate of volume flux through the pipe is given by

$$\int\int u_3\,dx\,dy=\frac{kh^2}{2}\int_0^{\xi_0}\int_0^{2\pi}(\cosh 2\xi-\cos 2\eta)\,u_3\,d\xi\,d\eta$$

$$=\frac{kh^2}{2}\left[\frac{2(Ek_1-M\hat{F})}{\sqrt{(\delta^2+4R)}\,\{(\delta^2+4R)^{1/2}+\delta\}}\right.$$

$$\times\left\{\pi\sinh 2\xi_0-\sum_{n=0}^{\infty}\left(4\pi^2\left(D_0^{2n}(q_1)\right)\right.\right.$$

$$\left.\int_0^{\xi_0}\left[D_0^{2n}(q_i)\cosh 2\xi-\tfrac{1}{2}D_2^{2n}(q_1)\right]Je_{2n}(\xi,q_1)\,d\xi)/Je_{2n}(\xi_0,q_i)\,Me_{2n}(q_i)\right]$$

$$+\frac{2(Ek_2-M\hat{F})}{\sqrt{(\delta^2+4R)}\,(\sqrt{(\delta^2+4R)}-\delta)}$$

$$\times\left\{\pi\sinh 2\xi_0-\sum_{n=0}^{\infty}\frac{4\pi^2 D_0^{2n}(-q_2)}{Je_{2n}(\xi_0,-q_2)\,Me_{2n}(-q_2)}\right.$$

$$\int_0^{\xi_0}\left[\left\{D_0^{2n}(-q_2)\cosh 2\xi-\tfrac{1}{2}D_2^{2n}(q_2')\right\}Je_{2n}(\xi,-q_2)\,d\xi\right\}$$

(equation continued on p. 1363)

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta e^{\alpha t}-\alpha e^{\beta t}}{(\alpha-\beta)}\;\frac{M\hat{F}+E(\delta-\gamma_{2n,m}^{2})}{\{(\gamma_{2n,m}^{2}-\delta)\,\gamma_{2n,m}^{2}-R\}}$$

$$\times\frac{\pi\int_{0}^{\xi_0}\frac{1}{2}\left[2D_0^{2n}(q_{2n,m})\cosh 2\xi-D_2^{2n}(q_{2n,m})\right]Je_{2n}(\xi,q_{2n,m})\,d\xi}{\int_{0}^{\xi_0}Je_{2n}^{2}(\xi,q_{2n,m})\,[Me(q_{2n,m})\cosh 2\xi-\Theta_{2n,m}]\,d\xi}\Bigg] \qquad \ldots(10.7)$$

If $R < R_c$ and $t \to \infty$ the steady state flux is given by first terms within curly brackets in eqn. (10.7).

(C) The mean temperature difference across a section of the pipe is given by

$$T_m=-\frac{\beta' h^2 a^2}{2\pi bR}\Bigg[\Bigg\{\frac{2k_2(Ek_1-M\hat{F})}{(\delta+\sqrt{(\delta^2+4R)})\sqrt{(\delta^2+4R)}}$$

$$\times\Bigg\{\pi\sinh 2\xi_0-\sum_{n=0}^{\infty}4\pi^2D_0^{2n}(q_1)$$

$$\times\frac{\int_0^{\xi_0}(D_0^{2n}(q_1)\cosh 2\xi-\frac{1}{2}D_2^{2n}(q_1))\,Je_{2n}(\xi,q_1)\,d\xi}{Je_{2n}(\xi_0,q_1)\,Me_{2n}(q_1)}\Bigg\}$$

$$+\frac{2k_1(Ek_2-M\hat{F})}{(\sqrt{(\delta^2+4R)}-\delta)(\delta^2+4R)^{1/2}}$$

$$\times\Bigg\{\pi\sinh 2\xi_0-\sum_{n=0}^{\infty}\frac{4\pi^2D_0^{2n}(-q_2)}{Je_{2n}(\xi_0,-q_2)\,Me_{2n}(-q_2)}$$

$$\times\int_0^{\xi_0}\left(D_0^{2n}(-q_2)\cosh 2\xi-\tfrac{1}{2}D_2^{2n}(-q_2)\right)Je_{2n}(\xi,q_2)\,d\xi\Bigg\}\Bigg\}$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta e^{\alpha t}-\alpha e^{\beta t}}{\beta-\alpha}\Bigg\{\frac{MF\gamma_{2n,m}^{2}-RE}{(\gamma_{2n,m}^{2}-\delta)\,\gamma_{2n,m}^{2}-R}\times$$

(*equation continued on p.* 1364)

$$\times \frac{\pi \int_0^{\xi_0} \frac{1}{2} \left[D_0^{2n}(q_{2n,m})\, 2\cosh 2\xi - D_2^{2n}(q_{2n,m}) \right] Je_{2n}(\xi, q_{2n,m})\, d\xi}{\int_0^{\xi_0} Je_{2n}^2(\xi, q_{2n,m}) \left[\cosh 2\xi\, Me_{2n}(q_{2n,m}) - \Theta_{2n,m} \right] d\xi} \Bigg\} \Bigg] \qquad \ldots(10.8)$$

when $t \to \infty$ and $R < R_c$ the steady state mean temperature difference is expressed by the expression in the curly brackets in eqn. (10.8).

(D) The Nusselt number in this case is obtained following the same method as in the case of circular cylinder which is a new result.

(E) Transition to circular cylinder : It can be easily verified that if $h \to 0$, $\xi \to \infty$, the elliptical cylinder degenerates into a circular cylinder and all the results mentioned above agree with the corresponding results of the circular pipe.

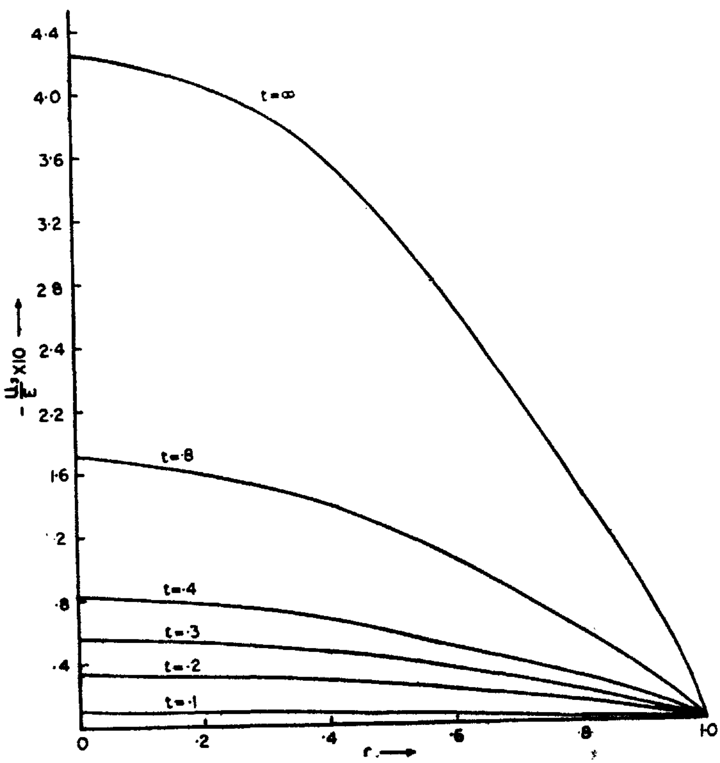

FIG. 1. Non-dimensional velocity profiles for falling convection for specified Rayleigh number $R = 10$ and coefficient of heat generation number $\delta = 3$ (circular pipe).

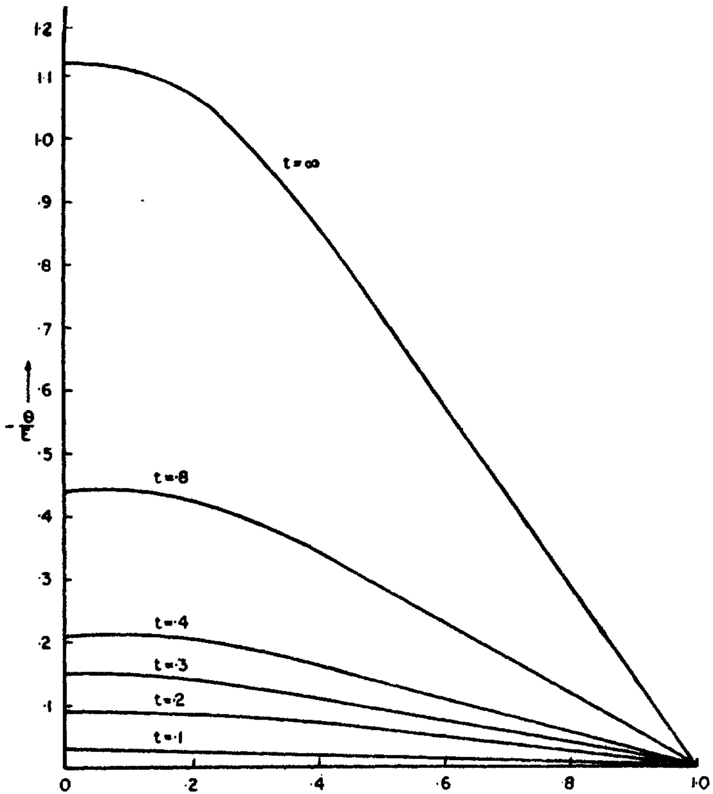

FIG. 2. Non-dimensional buoyancy profiles for specified Rayleigh number $R = 10$ and coefficient of source of heat generation number $\delta = 3$ (circular pipe).

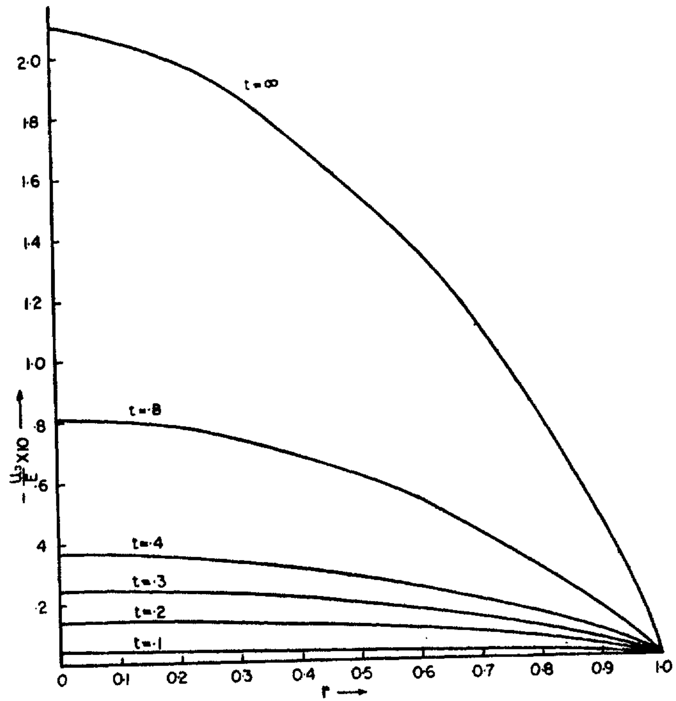

FIG. 3. Non-dimensional velocity profiles for falling convection for specified Rayleigh number $R = 5$ and coefficient of heat generation number $\delta = 4$ (circular pipe).

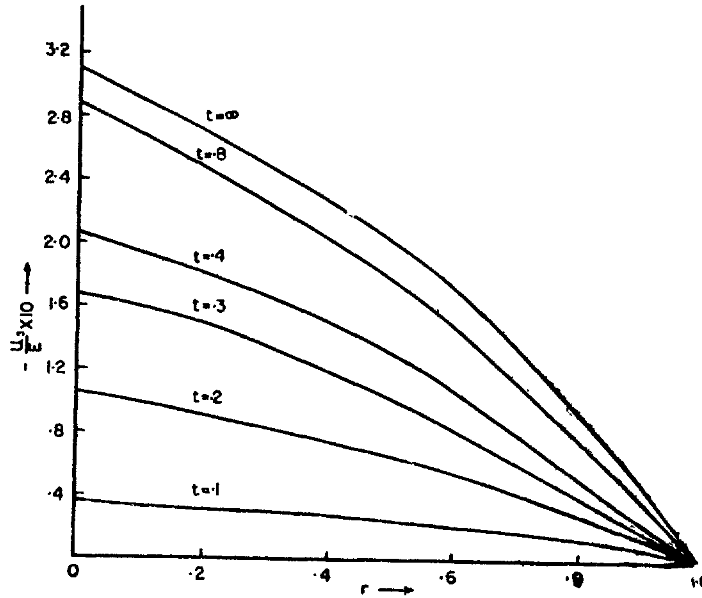

FIG. 4. Non-dimensional velocity profiles for falling convection for specified Rayleigh number $R=10$ and coefficient of heat generation number $\delta = -3$ (circular pipe).

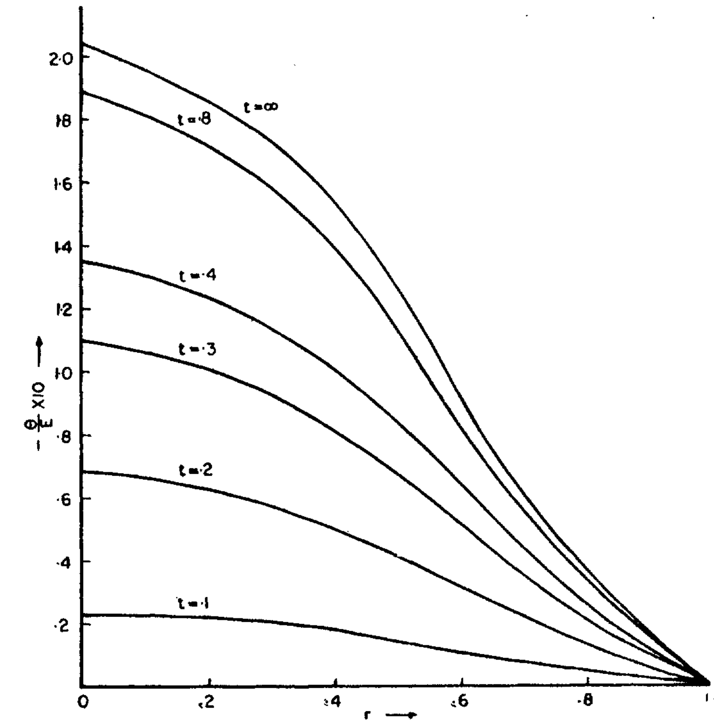

FIG. 5. Non-dimensional buoyancy profiles for specified Rayleigh number $R = 10$ and coefficient of source of heat generation number $\delta = -3$ (circular pipe).

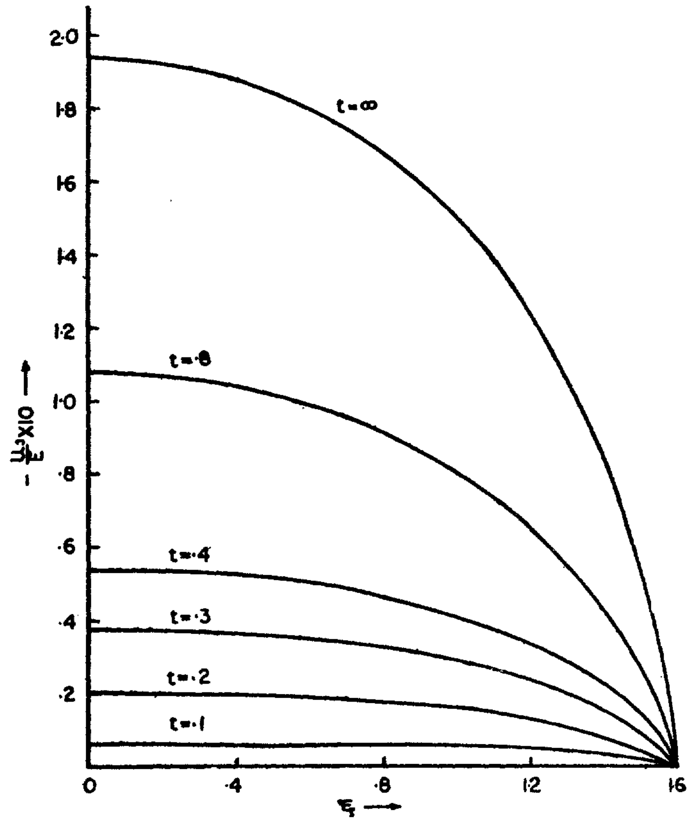

FIG. 6. Non-dimensional velocity profile for falling convection for specified Rayleigh number $R = 5$ and coefficient of heat generation number $\delta = 4$ (for elliptical pipe, $\eta = 0$).

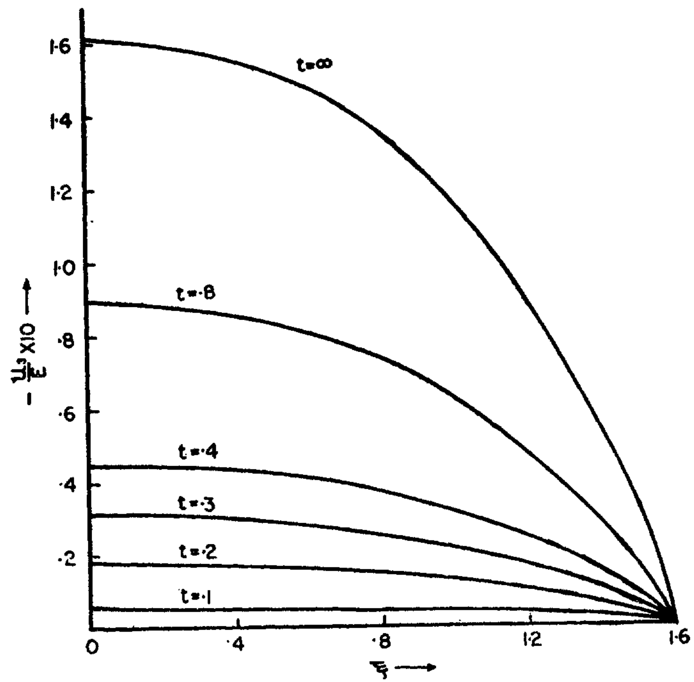

FIG. 7. Non-dimensional velocity profile for falling convection for specified Rayleigh number $R = 5$ and coefficient of heat generation number $\delta = 4$ (for elliptical pipe, $\eta = \pi/2$).

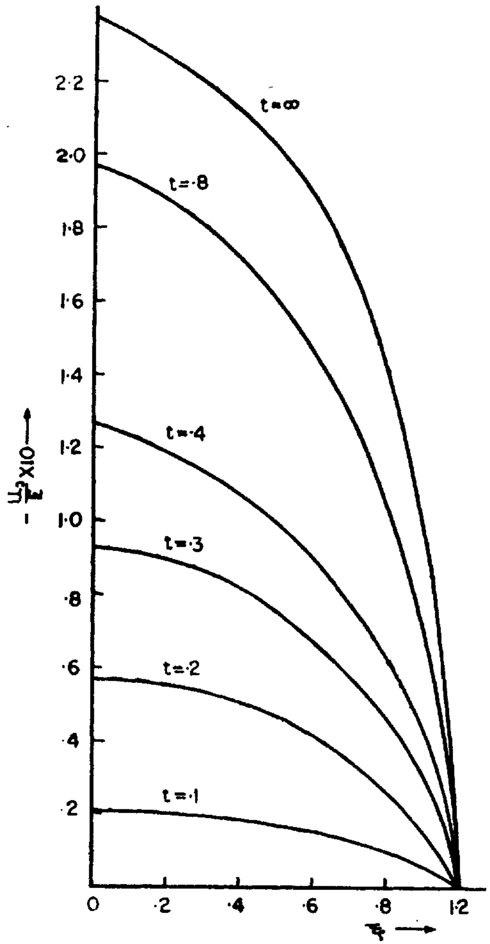

FIG. 8. Non-dimensional velocity profiles for falling convection for specified Rayleigh number $R = 10$ and coefficient of heat generation number $\delta = 3$ (elliptical pipe, $\eta = 0$).

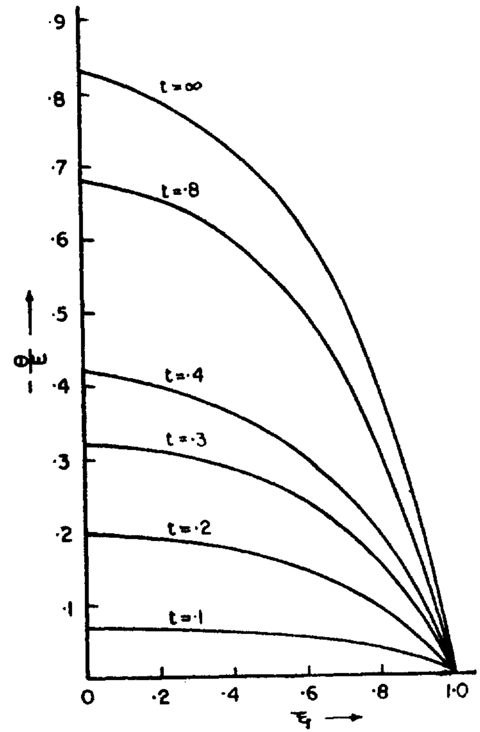

FIG. 9. Non-dimensional buoyancy profiles for specified Rayleigh number $R = 10$ and coefficient of source of heat generation number $\delta = 3$ (elliptical pipe, $\eta = 0$).

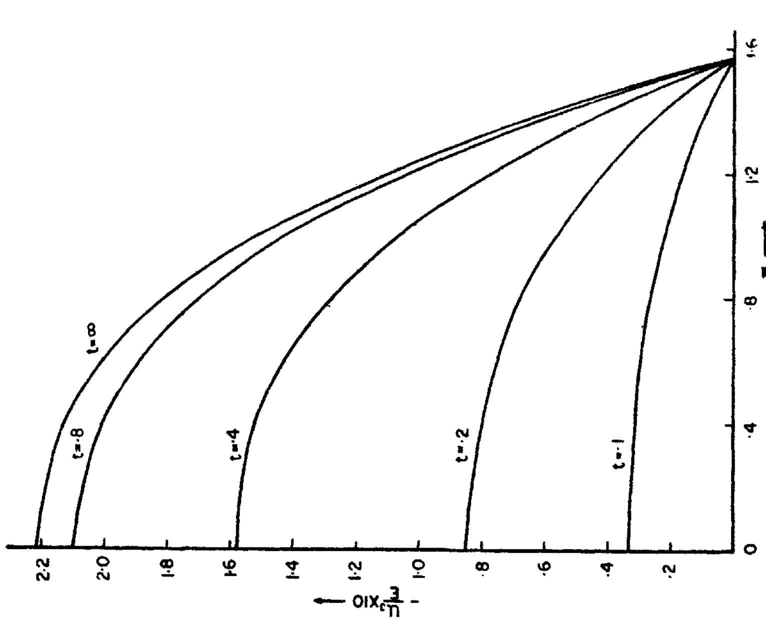

FIG. 10. Non-dimensional velocity profiles for falling convection for specified Rayleigh number $R = 10$ and coefficient of heat generation number $\delta = -3$ (elliptical pipe, $\eta = 0$).

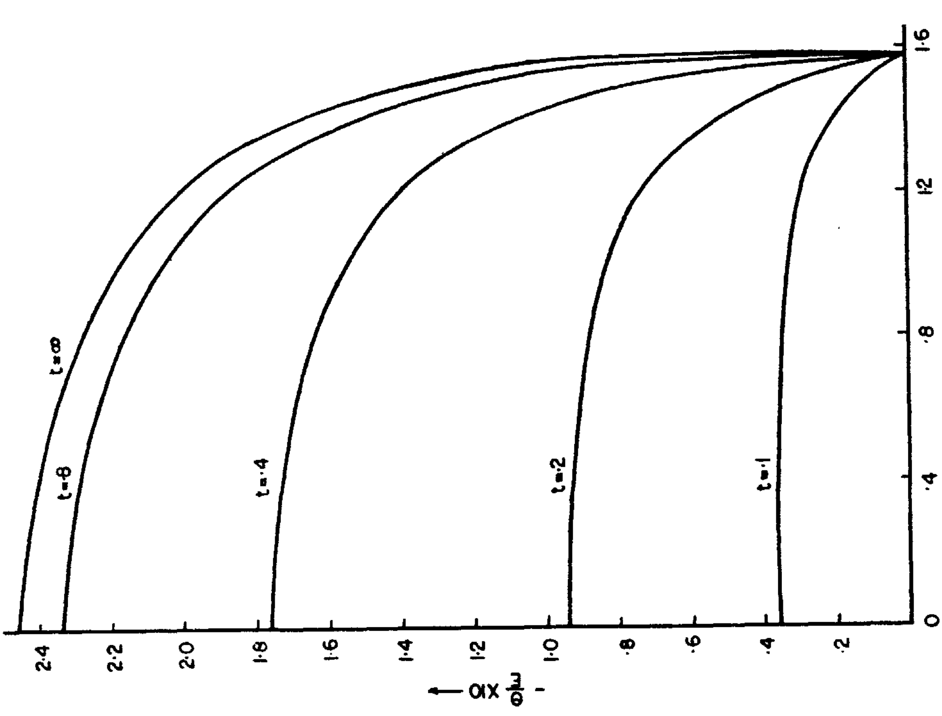

FIG. 11. Non-dimensional buoyancy profiles for specified Rayleigh number $R = 10$ and coefficient of source of heat generation number $\delta = -3$ (elliptical pipe, $\eta = 0$).

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APPENDIX A

Now we shall adopt the following procedure to sum the steady state terms. When we are dealing with the steady state problem, the equations are transformed into

$$\nabla^2 u_3 + \theta = E \quad \text{...(A.1)}$$

$$(\nabla^2 + \delta)\,\theta + R u_3 = -M\hat{F} \quad \text{...(A.2)}$$

where E and F are absolute constants. Multiplying (A.1) by k and adding we have

$$\nabla^2(\theta + k u_3) + (\delta + k)\,\theta + R u_3 = Ek - M\hat{F}$$

such that

$$\frac{1}{\delta + k} = \frac{k}{R}$$

or

$$k^2 + k\delta - R = 0$$

$$\text{i.e. } \frac{k_1}{k_2} = \frac{-\delta \pm \sqrt{(\delta^2 + 4R)}}{2}$$

hence we have the following two equations :

$$\nabla^2\phi + \frac{\delta + \sqrt{(\delta^2 + 4R)}}{2}\,\phi = Ek_1 - M\hat{F}$$

$$\nabla^2\phi_1 - \frac{(\sqrt{(\delta^2 + 4R)} - \delta)}{2}\,\phi_1 = Ek_2 - M\hat{F}$$

where

$$\phi = \theta + k_1 u_3,\ \phi_1 = \theta + k_2 u_3.$$

The boundary conditions give $\phi = \theta = \phi_1$ on $\xi = \xi_0$ or $r = 1$ as the case may be. The solutions of these equations under these boundary conditions in elliptical coordinates are

$$u_3 = \frac{2(Ek_1 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} + \delta)\sqrt{(\delta^2 + 4R)}}$$

$$\times \left\{1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q_1)\, Je_{2n}(\xi, q_1)\, Se_{2n}(\eta, q_1)}{Je_{2n}(\xi_0, q_1)\, Me_{2n}(q_1)}\right\}$$

$$+ \frac{2(Ek_2 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} - \delta)\sqrt{(\delta^2 + 4R)}}$$

$$\times \left\{1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(-q_2)\, Je_{2n}(\xi, q_2)\, Se_{2n}(\eta - q_2)}{Je_{2n}(\xi_0, -q_2)\, Me_{2n}(-q_2)}\right\} \qquad \text{...(A.3)}$$

where

$$\theta = \frac{2k_2(Ek_1 - M\hat{F})}{(\delta + \sqrt{(\delta^2 + 4R)})\sqrt{(\delta^2 + 4R)}}$$

$$\times \left\{1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(q_1)\, Je_{2n}(\xi, q_1)\, Se_{2n}(\eta, q_1)}{Je_{2n}(\xi_0, q_1)\, Me_{2n}(\eta, q_1)}\right\}$$

$$+ \frac{2k_1(Ek_2 - M\hat{F})}{(\sqrt{(\delta^2 + 4R)} - \delta)\sqrt{(\delta^2 + 4R)}}$$

$$\times \left\{1 - \sum_{n=0}^{\infty} \frac{2\pi D_0^{2n}(-q_2)\, Je_{2n}(\xi, q_2)\, Se_{2n}(\eta, -q_2)}{Je_{2n}(\xi_0, -q_2)\, Me_{2n}(-q_2)}\right\}. \qquad \text{...(A.4)}$$

Now we use the transform defined in §6, so that we may be able to find the transform of u_3 and θ and their inversions. Multiplying (A.1) and (A.2) by $Se_{2n}(\eta, q_{2n,m}) \times Je_{2n}(\xi, q_{2n,m})$ where $q_{2n,m}$ is the mth positive root of $Je_{2n}(\xi_0, q) = 0$ and integrating with respect to η from 0 to 2π and with respect to ξ between 0 and ξ_0, we have as in (6.1)

$$\gamma_{2n,m}^2 \bar{u}_3 = \bar{\theta} - \bar{E}$$

$$\gamma_{2n,m}^2 \bar{\theta} - \delta\bar{\theta} = R\bar{u}_3 + M\bar{\hat{F}}. \qquad \text{...(A.5)}$$

On solving these equations, we get

$$\bar{u}_3 = \frac{M\bar{\hat{F}} - \bar{E}(\gamma_{2n,m}^2 - \delta)}{(\gamma_{2n,m}^2 - \delta)\gamma_{2n,m}^2 - R}, \quad \bar{\theta} = \frac{M\bar{F}\gamma_{2n,m}^2 - R\bar{E}}{(\gamma_{2n,m}^2 - \delta)\gamma_{2n,m}^2 - R}. \qquad \text{...(A.6)}$$

It is quite obvious that the value of $\bar{\theta}$ and $\bar{u}_3$ found above satisfy (A.1) and (A.2) hence the transform of u_3 and θ as given by (A.6) above as the same as in §8. Hence inversion gives the sum of the series for the steady state.