ELECTROMAGNETIC FIELDS OF EMBEDDING CLASS ONE

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(Communicated by F. C. Auluck, F.N.A.)

(*Received* 5 *July* 1973; *after revision* 27 *June* 1974)

Considering the eigenvalues and eigenvectors of the second fundamental form b_{ij}, it has been shown that a space-time of class one can describe an electromagnetic energy distribution if and only if the elementary divisors of b_{ij} are not simple and such a field is necessarily null and is not conformally flat. A canonical form for b_{ij} describing a null electromagnetic field is obtained. Lastly, a null field has been constructed starting from a space-time, of embedding class one.

INTRODUCTION

It is well known that pure gravitational fields, satisfying $R_{ij} = 0$, cannot be embedded in a Euclidean space of five dimensions. The class of such fields is always more than one. Among the distributions, many perfect fluids like Schwarzchild's interior solution, Friedmann cosmological model etc. are known to be of class one. Recently two distinct types of perfect fluid distributions of class one, depending on the nature of eigenvalues and eigenvectors of the second fundamental form b_{ij} has been investigated (Pandey and Gupta 1970). Pandey and Kansal (1968) have shown that the spherically symmetric electromagnetic field of class one does not exist.

In the present paper we have investigated the possibility of electromagnetic fields being represented by a space-time of class one in general. However, this problem has already been tackled by Collinson (1968), with different angles. Here we have shown that an electromagnetic distribution of energy is compatible with a class one conditions if and only if the elementary divisors of the second fundamental form b_{ij} are not simple and such fields are necessarily null and are not conformally flat. In this case b_{ij} admits a canonical form

$$b_{ij} = -\lambda v_i v_j + \sigma u_i u_j,\ v_i v^i = -1,\ u_i u^i = 0,\ u_i v^i = 0$$

where λ is only non-zero eigenvalue of b_{ij} and σ is a scalar.

A CANONICAL FORM OF b_{ij} FOR CLASS ONE ELECTROMAGNETIC FIELD

When a Riemannian space-time is of class one, there exists a symmetrical tensor b_{ij} satisfying the following conditions

$$\left.\begin{aligned} R_{hijk} &= e(b_{hj}\, b_{ik} - b_{hk}\, b_{ij}),\ e = \mp 1 \\ b_{ij;k} - b_{ik;j} &= 0 \end{aligned}\right\} \quad \ldots(1)$$

Pandey and Kansal (1969) proved the impossibility of class one electromagnetic field in the case when the elementary divisors of b_{ij} are simple. However, this theorem does not cover the cases when these are multiple.

If one, or more of the elementary divisors of b_{ij} are multiple and real at a point, it means that g_{ij} and b_{ij} are one of the following types (Eisenhart 1960).

$$\left.\begin{aligned} \textit{Type 1}: g_{14} = 1,\ g_{33} = -k_3,\ g_{22} = -k_2,\ k_2 > 0,\ k_3 > 0 \\ b_{11} = k_1,\ b_{14} = \lambda_1,\ b_{33} = -\lambda_3 k_3,\ b_{22} = -\lambda_2 k_2. \end{aligned}\right\} \quad \ldots(2)$$

Other b_{ij} and g_{ij} are identically zero. The elementary divisors are

$$(\lambda - \lambda_1)^2,\ (\lambda - \lambda_3) \text{ and } (\lambda - \lambda_2).$$

$$\left.\begin{aligned} \textit{Type 2}: g_{14} = 1,\ g_{33} = -k_3,\ g_{22} = -k_2,\ k_2 > 0,\ k_3 > 0 \\ b_{14} = \lambda_1,\ b_{34} = 1,\ b_{33} = -\lambda_3 k_3,\ b_{22} = -k_2\lambda_2 \end{aligned}\right\} \quad \ldots(3)$$

The elementary divisors are $(\lambda - \lambda_1)^3$, $(\lambda - \lambda_2)$. Two more types are possible, but they do not satisfy the relativistic requirements.

The components of the Ricci tensor for Type 1 calculated with the help of first equation of (1) and (2) are furnished below,

$$\left.\begin{aligned} R_{11} &= e\, k_1(\lambda_2 + \lambda_3) \\ R_{22} &= e\, k_2\lambda_2(2\lambda_1 + \lambda_3) \\ R_{33} &= e\, k_3\lambda_3(2\lambda_1 + \lambda_2) \\ R_{14} &= e\, \lambda_1(\lambda_1 + \lambda_2 + \lambda_3). \end{aligned}\right\} \quad \ldots(4)$$

In consequence of (4) and Rainich conditions

$$R = 0,\ R_i^m R_m^j = \sigma\, \delta_i^j,\ \sigma = \frac{1}{4} R^{mn} R_{mn}\ (\sigma \geqslant 0)$$

$$v_{i,j} - v_{j,i} = 0$$

$$v_k = (\sqrt{-g}\, \epsilon_{ilmn}\ R^{mi;l} R_k^n)/R_{ij} R^{ij}. \quad \ldots(5)$$

We have two distinct cases,

$$\sigma = 0,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 \neq 0 \quad \ldots(6a)$$

$$\sigma = 0,\ \lambda_1 = 0,\ \lambda_3 = 0,\ \lambda_2 \neq 0. \quad \ldots(6b)$$

The eigenvectors, associated with the eigenvalues occurring in (6*a*) and (6*b*) are expressible in the form,

$$v_i = (0, 0, -\sqrt{k_3}, 0), \quad u_i = (1, 0, 0, 0) \qquad \text{...(7a)}$$

and $$v_i = (0, -\sqrt{k_2}, 0, 0), \quad u_i = (1, 0, 0, 0) \qquad \text{...(7b)}$$

where in both the cases

$$v_i v^i = -1, u_i u^i = 0 \text{ and } u_i v^i = 0.$$

Similar analysis in case of (3) leads to a flat space time. Hence we conclude that only Type 1 can describe an electromagnetic distribution and the field is always null. Moreover (7*a*) and (7*b*) give rise to an interesting canonical form for b_{ij}, in Type 1 i.e.

$$b_{ij} = -\lambda v_i v_j + \sigma u_i u_j \qquad \text{...(8)}$$

where λ is only nonzero eigenvalue of b_{ij} and σ is some scalar.

The eqn. (8) with (1) give rise to

$$R_{ij} = \mp \sigma\lambda u_i u_j \qquad \text{...(9)}$$

which is a characteristic feature of null fields.

Now the non-vanishing components of Weyl curvature tensor in case of (6*a*) are given to be

$$C_{1313} = -\frac{3}{2} e\, k_1 k_3 \lambda_3 \text{ and } C_{1212} = -\frac{e}{2} k_1 k_2 \lambda_3.$$

It leads to the conclusion that space-time corresponding to the (6*a*) or (6*b*) cannot be conformally flat unless $\lambda_3 = 0$. The latter demands the flatness of the space.

A New Null Field of Class One

The most general line element of class one can be expressed as

$$ds^2 = -(dx')^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 \mp d\psi^2$$

where $\psi = \psi(x^1, x^2 . x^3, x^4)$. ...(10)

Considering negative sign, the metric potentials, and components of Ricci-tensor are given as

$$g^{\mu\nu} = \eta_{\mu\nu} - \psi_\mu\psi_\nu / \mid g_{ij} \mid, \; g^{\mu 4} = \psi_\mu \psi_4 / \mid g_{ij} \mid \qquad \text{...(11a)}$$

$$R_{ik} = -g^{hj} (\psi_{hk}\psi_{ij} - \psi_{hj}\psi_{ik}) / \mid g_{ij} \mid \qquad \text{...(11b)}$$

where

$$\psi_{ij} = \frac{\partial^2\psi}{\partial x^i \partial x^j}, \; \eta_{ij} = \text{diag}(-1, -1, -1, +1).$$

In order to have a function ψ, such that (10) may describe a null electromagnetic field distribution let us assume, the vector u_i in (9) to be of the form

$$u_i = (0, 0, 1, \mp 1) \qquad \text{...(12)}$$

Since it is a null vector we have

$$g^{ij}\, u_i u_j = 0 \qquad \text{...(13)}$$

which gives an account of (11*a*) and (12)

$$\psi_3 \pm \psi_4 = 0. \qquad \text{...(14)}$$

A convenient solution of (14) can be written as

$$\psi = \phi(x^1, x^2)\, f(x^3 \mp x^4) \qquad \text{...(15)}$$

where ϕ and f are arbitrary functions.

Now (15) with reference to (9) and (11*b*) gives

$$\phi_{12}^2 - \phi_{11}\phi_{22} = 0, \quad \phi_{12}\phi_1 - \phi_{11}\phi_2 = 0$$

and $$\phi_2\phi_{12} - \phi_{22}\phi_1 = 0 \qquad \text{...(16)}$$

when $$\phi_{ij} = \frac{\partial^2\phi}{\partial x^i \partial x^j}.$$

Solution to above equations has been found to be

$$\phi = \phi(\alpha x^1 + \beta x^2) \qquad \text{...(17)}$$

where α and β are arbitrary constants.

Hence the geometrical property of the required null field distribution is given by the metric (10) where

$$\psi = \phi(\alpha x^1 + \beta x^2)\, f(x^3 \mp x^4). \qquad \text{...(18)}$$

Conclusions

It is established that class one electromagnetic field may exist if and only if the elementary divisors of the second fundamental form b_{ij} are not simple and such a field is necessarily null.

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