ON THE BEHAVIOUR OF FOURIER SINE TRANSFORMS NEAR THE ORIGIN

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The author has proved a more general form of Hartman and Wintner's theorem (1951) in the present paper.

§1. It has been proved by Hartman and Wintner (1951) that if $f(x) \geqslant 0$ for all positive x and $f(x)$ tends steadily to zero as $x \to \infty$, then

$$F(t) = \int_{+0}^{\infty} \sin xt\, f(x)\, dx$$

exists for $0 < t < \infty$ and

$$\operatorname*{limit}_{t \to 0} \frac{F(t)}{t} = \int_{+0}^{\infty} x f(x)\, dx$$

provided that

$$\int_{+0}^{1} x f(x)\, dx < \infty.$$

The object of this paper is to prove a more general form of this theorem.

Theorem — If $f(x)$ and $\psi(x)$ are positive for all positive large x, and $\psi(x)$ steadily increases to infinity as x tends to infinity such that

$$\int_{a}^{\infty} \frac{x\, dx}{\psi(x)} < \infty \qquad \text{...(1.1)}$$

for a sufficiently large a and if

for all x and x_1, with $a \leqslant x \leqslant x_1$

$$f(x_1) < f(x) + \frac{k}{\psi(x)} \qquad \text{...(1.2)}$$

k being a positive constant, then

$F(t)$ exists for $0 < t < \infty$ and

$$\lim_{t \to 0} \frac{F(t)}{t} = \int_{+0}^{\infty} x f(x)\, dx \qquad \ldots(1.3)$$

provided that $f(x) \to 0$ as $x \to \infty$ and

$$\int_{+0}^{1} x \mid f(x) \mid dx < \infty \qquad \ldots(1.4)$$

§2. We shall require the following lemmas for the proof of our theorem.

Lemma 1 — If $\int_{a}^{\infty} \frac{x}{\psi(x)}\, dx < \infty$ where $\psi(x)$ steadily increases to ∞, then for fixed t,

$$\operatorname*{Lim}_{r \to \infty} \sum_{n=r}^{\infty} \frac{1}{\psi\left(\frac{n\pi}{t}\right)} = 0.$$

Proof : For a fixed t we choose C such that

$$\frac{(C-1)\pi}{t} > a.$$

Then

$$\int_{a}^{\infty} \frac{x}{\psi(x)}\, dx = \int_{a}^{(C-1)\pi/t} \frac{x}{\psi(x)}\, dx + \int_{(C-1)\pi/t}^{\infty} \frac{x}{\psi(x)}\, dx$$

$$= I_1 + I_2, \text{ say.}$$

Now

$$I_2 \geqslant \frac{(C-1)\pi}{t} \int_{(C-1)\pi/t}^{\infty} \frac{dx}{\psi(x)}$$

$$= \frac{(C-1)\pi}{t} \sum_{n=C}^{\infty} \int_{(n-1)\pi/t}^{n\pi/t} \frac{dx}{\psi(x)}$$

(*equation continued on p.* 1461)

$$> \frac{(C-1)\pi}{t} \sum_{n=C}^{\infty} \frac{1}{\psi \frac{(n\pi)}{t}} \left(\frac{n\pi}{t} - \frac{(n-1)\pi}{t} \right)$$

$$= \frac{\pi^2(C-1)}{t^2} \sum_{n=C}^{\infty} \frac{1}{\psi\left(\frac{n\pi}{t}\right)}.$$

Since $\quad I_2 = O(1),$

$$\sum_{n=C}^{\infty} \frac{1}{\psi\left(\frac{n\pi}{t}\right)} < \infty \cdot$$

Hence $\quad \lim\limits_{r\to\infty} \sum\limits_{n=r}^{\infty} \frac{1}{\psi\left(\frac{n\pi}{t}\right)} = 0,$

which completes the proof of the lemma.

Lemma 2 — If $\int\limits_{a}^{\infty} \frac{x\,dx}{\psi(x)} < \infty$ where $\psi(x)$ is as defined in the theorem, then

$$\lim_{t\to 0} \frac{1}{t^2} \sum_{r=1}^{\infty} \frac{1}{\psi\left(\frac{r\pi}{t}\right)} = 0.$$

PROOF : Since $\int\limits_{a}^{\infty} \frac{x\,dx}{\psi(x)} < \infty$

we have

$$\frac{1}{4} \int_{2a}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)} < \infty,$$

i.e., $\quad \int\limits_{2a}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)} < \infty,$

and

$$\int_{2a}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)}\,dy = \int_{2a}^{\pi/t} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)} + \int_{\pi/t}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)} < \infty.$$

Hence

$$\lim_{t\to 0} \int_{\pi/t}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)} = 0.$$

Now

$$\int_{\pi/t}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)} > \sum_{r=1}^{\infty} \int_{(2r-1)\pi/t}^{2r\pi/t} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)}$$

$$> \sum_{r=1}^{\infty} \frac{1}{\psi\left(\frac{r\pi}{t}\right)} \int_{(2r-1)\pi/t}^{2r\pi/t} y\,dy$$

$$= \sum_{r=1}^{\infty} \frac{1}{\psi\left(\frac{r\pi}{t}\right)} \frac{\pi^2}{2t^2} (4r^2 - 4r^2 + 4r - 1)$$

$$\geqslant \frac{3\pi^2}{2t^2} \sum_{r=1}^{\infty} \frac{1}{\psi\left(\frac{r\pi}{t}\right)}$$

and so $\displaystyle \frac{1}{t^2} \sum_{r=1}^{\infty} \frac{1}{\psi\left(\frac{r\pi}{t}\right)} < \frac{2}{3\pi^2} \int_{\pi/t}^{\infty} \frac{y\,dy}{\psi\left(\frac{y}{2}\right)}.$

Hence $\displaystyle \lim_{t\to 0} \frac{1}{t^2} \sum_{r=1}^{\infty} \frac{1}{\psi\left(\frac{r\pi}{t}\right)} = 0.$

§3. *Proof of Theorem* — Let X be any fixed number greater than a and if r is an odd positive integer such that

$$(r-1)\pi \leqslant tX < r\pi, \quad t > 0$$

then we have

$$\int_{X}^{r\pi/t} f(x)\sin xt\,dx \leqslant \frac{1+\cos Xt}{t}\left\{\left(f(X)+\frac{k}{\psi(X)}\right)\right\},$$

$$\int_{r\pi/t}^{(r+1)\pi/t} f(x)\sin xt\,dx \leqslant \int_{r\pi/t}^{(r+1)\pi/t}\left\{f\left(\frac{(r+1)\pi}{t}\right)-\frac{k}{\psi\left(\frac{r\pi}{t}\right)}\right\}\sin xt\,dx$$

$$= -\frac{2}{t}f\left(\frac{(r+1)\pi}{t}\right)+\frac{2k}{t\psi\left(\frac{r\pi}{t}\right)}$$

$$\int_{(r+1)\pi/t}^{(r+2)\pi/t} f(x)\sin xt\,dx \leqslant \int_{(r+1)\pi/t}^{(r+2)\pi/t}\left\{f\left(\frac{(r+1)\pi}{t}+\frac{k}{\psi\left(\frac{(r+1)\pi}{t}\right)}\right\}\sin xt\,dx$$

$$= \frac{2}{t}f\left(\frac{(r+1)\pi}{t}\right)+\frac{2k}{t\psi\left(\frac{(r+1)\pi}{t}\right)}$$

and so on.

Similarly if r is an even positive integer and $(r-1)\pi \leqslant tX < r\pi$, we have

$$\int_{X}^{r\pi/t} f(x)\sin xt\,dx$$

$$\leqslant -\frac{1-\cos Xt}{t}\left\{f\left(\frac{r\pi}{t}\right)-\frac{k}{\psi(X)}\right\}$$

$$= -\frac{2}{t}f\left(\frac{r\pi}{t}\right)+\frac{1+\cos Xt}{t}f\left(\frac{r\pi}{t}\right)$$

$$+\frac{1-\cos Xt}{t}\,\frac{k}{\psi(X)},$$

$$\int_{r\pi/t}^{(r+1)\pi/t} f(x)\sin xt\,dx \leqslant \frac{2}{t}f\left(\frac{r\pi}{t}\right)+\frac{2k}{t\psi\left(\frac{r\pi}{t}\right)}$$

and so on.

Hence by virtue of Lemma 1 and the condition that $f(x) \to 0$ and $\psi(x) \to \infty$ as $x \to \infty$ we have

$$\lim_{X\to\infty} \int_X^\infty f(x) \sin xt\, dx \leqslant 0.$$

Again if r is an odd positive integer such that

$$(r-1)\pi \leqslant tX < r\pi, \quad t > 0,$$

$$\int_X^{r\pi/t} f(x) \sin xt\, dx \geqslant \int_X^{r\pi/t} \left\{ f\left(\frac{r\pi}{t}\right) - \frac{k}{\psi(x)} \right\} \sin xt\, dx$$

$$\geqslant \frac{2}{t} f\left(\frac{r\pi}{t}\right) - \frac{1-\cos Xt}{t} f\left(\frac{r\pi}{t}\right) - kt \int_X^{r\pi/t} \frac{x\, dx}{\psi(x)}$$

$$\int_{r\pi/t}^{(r+1)\pi/t} f(x) \sin xt\, dx \geqslant -\frac{2}{t} f\left(\frac{r\pi}{t}\right) - \frac{2k}{t\psi\left(\frac{r\pi}{t}\right)}$$

$$\int_{(r+1)\pi/t}^{(r+2)\pi/t} f(x) \sin xt\, dx \geqslant \int_{(r+1)\pi/t}^{(r+2)\pi/t} \left\{ f\left(\frac{(r+2)\pi}{t}\right) - \frac{k}{\psi(x)} \right\} \sin xt\, dx$$

$$\geqslant \frac{2}{t} f\left(\frac{(r+2)\pi}{t}\right) - kt \int_{(r+1)\pi/t}^{(r+2)\pi/t} \frac{x\, dx}{\psi(x)}$$

$$\int_{(r+2)\pi/t}^{(r+3)\pi/t} f(x) \sin xt\, dx \geqslant -\frac{2}{t} f\left(\frac{(r+2)\pi}{t}\right) - \frac{2k}{t\psi\left(\frac{(r+2)\pi}{t}\right)}$$

and so on.

Therefore

$$\int_X^\infty f(x) \sin xt\, dx \geqslant -\frac{2k}{t} \sum_{n=r}^{\infty} \frac{1}{\psi\left(\frac{n\pi}{t}\right)} - kt \int_X^\infty \frac{x\, dx}{\psi(x)}$$

$$-\frac{(1-\cos Xt)}{t} f\left(\frac{r\pi}{t}\right).$$

Hence by virtue of Lemma 1 and the condition that

$$\int_a^\infty \frac{x}{\psi(x)}\,dx < \infty$$

and that $f(x) \to 0$ as $X \to \infty$.

We have

$$\lim_{X\to\infty} \int_X^\infty f(x) \sin xt\,dx \geqslant 0,$$

so that

$$\lim_{X\to\infty} \int_X^\infty f(x) \sin xt\,dx = 0$$

since $\lim_{X\to\infty} \int_X^\infty f(x) \sin xt\,dx = 0$ and by hypothesis

$\int_{+0}^1 xf(x)\,dx$ is absolutely convergent it follows that

$$F(t) = \int_{+0}^\infty f(x) \sin xt\,dx$$

is convergent for $0 < t < \infty$.

Now choose X sufficiently large and keeping it fixed we have

$$F(t) = \int_0^\infty f(x) \sin xt\,dx$$

$$= \int_0^X f(x) \sin xt\,dx + \int_X^{\pi/t} f(x) \sin xt\,dx$$

$$+ \int_{\pi/t}^\infty f(x) \sin xt\,dx$$

Therefore

$$\overline{\lim_{t\to 0}} \frac{F(t)}{t} \leqslant \overline{\lim_{t\to 0}} \int_0^X f(x) \frac{\sin xt}{t}\, dx$$

$$+ \overline{\lim_{t\to 0}} \int_X^{\pi/t} f(x) \frac{\sin xt}{t}\, dx$$

$$+ \overline{\lim_{t\to 0}} \frac{1}{t^2} \sum_{n=1}^{\infty} \frac{1}{\psi\left(\frac{n\pi}{t}\right)}$$

$$\leqslant \int_{+0}^{\infty} x f(x)\, dx$$

since the last limit is zero by virtue of Lemma 2.

Again

$$\int_X^{\pi/t} f(x) \frac{\sin xt}{t}\, dx \geqslant \int_X^{\pi/t} \left\{ f\left(\frac{\pi}{t}\right) - \frac{k}{\psi(x)} \right\} \frac{\sin xt}{t}\, dx$$

$$\geqslant \frac{2}{t^2} f\left(\frac{\pi}{t}\right) - \frac{1-\cos Xt}{t} f\left(\frac{\pi}{t}\right) - k \int_X^{\pi/t} \frac{x\, dx}{\psi(x)}$$

$$\int_{\pi/t}^{2\pi/t} f(x) \sin xt\, dx \geqslant -\frac{2}{t^2} f\left(\frac{\pi}{t}\right) - \frac{2k}{t^2 \psi\left(\frac{\pi}{t}\right)}$$

proving thereby that when x is sufficiently large and fixed

$$\underline{\lim_{t\to 0}} \int_X^{\infty} f(x) \frac{\sin xt}{t}\, dx \geqslant -k \int_X^{\infty} \frac{x\, dx}{\psi(x)}.$$

Consequently we have

$$\underline{\lim_{t\to 0}} \frac{F(t)}{t} \geqslant \int_{+0}^{X} x f(x)\, dx - k \int_X^{\infty} \frac{x\, dx}{\psi(x)}.$$

Since

$$\int_a^\infty \frac{x\,dx}{\psi(x)} < \infty.$$

Hence

$$\int_X^\infty \frac{x\,dx}{\psi(x)} \to 0, \quad \text{as } X \to \infty.$$

Therefore,

$$\varliminf_{t\to 0} \frac{F(t)}{t} \geqslant \int_{+0}^\infty xf(x)\,dx,$$

which completes the proof of the theorem.

Reference

Hartman, Philip, and Wintner, Aurel (1951). On the behaviour of Fourier sine transforms near the origin. *Proc. Am. math. Soc.*, **2**, 398-400.