SOME FUNCTION SPACE TOPOLOGIES FOR MULTIFUNCTIONS

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The paper concerns with generalizations of some function space topologies to include a family of multifunctions from one space to another. Concepts of near topologies, graph topologies, σ-topologies etc., for multifunctions have been introduced and discussed in details. Results concerning their separation properties and their relationship with compact open topology and pointwise convergence topology for multifunctions (introduced by Smithson) are also included.

1. INTRODUCTION

Various function space topologies for the family of single-valued mappings from one topological space to another have been defined and studied by many authors. Not much is known about function space topologies for multifunctions. Recently, Smithson (1971, 1973) has initiated the study of function spaces for multifunctions. He has generalized the concepts of pointwise convergence topology, compact open topology and the topology of uniform convergence for multifunctions. In the present paper, we generalize some other function space topologies, such as the graph topologies introduced by Naimpally (1966) and Naimpally and Pareek (1970), the σ-topology introduced by Arens and Dugundji (1951) and the near topology introduced by Irudayanathan (1967) to include the class of all multifunctions. Section 2 deals with compact open topology. In sections 3, 4 and 5 respectively graph topologies, σ-topology and near topology for multifunctions have been introduced and studied.

Throughout, X and Y will denote topological spaces. Y^X and Y^{mX} will denote respectively the class of all single-valued mappings and the class of all multifunctions from X to Y. The following definitions and notations will be used regarding multifunctions.

Definition 1.1 — A mapping f from X to Y which maps each point of X to a subset of Y is called a multifunction. For any subset A of X, $f(A) = \bigcup_{x \in A} f(x)$. For any subset B of Y, $f^+(B)$ is the set $\{x \in X : f(x) \subset B\}$ and $f^-(B)$ is the set $\{x \in X : f(x) \cap B \neq \phi\}$.

Definition 1.2 — A multifunction $f : X \to Y$ is said to be one-one if for every pair of distinct points x, y of X $f(x) \cap f(y) = \phi$.

Definition 1.3 — Let $f : X \to Y$ be a multifunction from X to Y. Then f is said to be upper semi-continuous (u.s.c.) at a point $x \in X$ if for each open subset U of Y containing the set $f(x)$ there exists an open subset V of X containing x such that $f(x_0) \subset U$ for all $x_0 \in V$. Multifunction f is called lower semi-continuous (l.s.c.) at $x \in X$ if for each open subset U of Y such that $f(x) \cap U \neq \phi$, there exists an open subset V of X containing x such that $f(x_0) \cap U \neq \phi$ for all $x_0 \in V$. It can easily be verified that f is u.s.c. (l.s.c.) if and only if $f^+(U)$ ($f^-(U)$) is open for each open subset U of Y. Further, f is said to be continuous at a point $x \in X$ if it is both u.s.c. and l.s.c. at x and f is continuous if it is continuous at every point of X. Thus f is continuous if and only if $f^-(U)$ and $f^+(U)$ are open in X for every open subset U of Y.

Definition 1.4 — A multifunction $f : X \to Y$ is said to be point-closed (point-compact, respectively) if $f(x)$ is a closed (compact, respectively) subset of Y for each $x \in X$. Also f is said to be closed (compact, respectively) if for each closed (compact) subset A of X, $f(A)$ is closed (compact) in Y.

Definition 1.5 — If $f : X \to Y$ is a multifunction, then the graph of f is the subset $\cup \{\{x\} \times f(x) : x \in X\}$ of $X \times Y$. Graph of f is denoted by $G(f)$.

Definition 1.6 : (Smithson 1973) — Let X be any set and Y be any topological space. For each $x \in X$, let $\Pi_x : Y^{mX} \to Y$ be a multifunction defined as $\Pi_x(f) = f(x)$ for all $f \in Y^{mX}$. Mappings Π_x are called projection mappings. Then, as in the case of single-valued mappings, the smallest topology on Y^{mX} for which each projection mapping is continuous is called the point-open topology or the topology of pointwise convergence (abbreviated as p.c. topology). We shall denote the topology of pointwise convergence by $\mathcal{P}$. Thus $\mathcal{P}$ is generated by a sub-base consisting of all sets of the form $\{f \in Y^{mX} : f(x) \subset U\}$ and $\{f \in Y^{mX} : f(x) \cap U \neq \phi\}$, where $x \in X$ and U is an open subset of Y.

2. Compact Open Topology

Definition 2.1 . (Smithson 1973) — Let K be a compact subset of X and U be an open subset of Y. Let $W_1(K, U) = \{f \in Y^{mX} : f(x) \cap U \neq \phi$ for all $x \in K\}$ and $W_2(K, U) = \{f \in Y^{mX} : f(K) \subset U\}$. The topology $\mathcal{C}$ on Y^{mX} generated by all sets $W_1(K, U)$ and $W_2(K, U)$ is then called the compact open topology for Y^{mX}. Obviously, $\mathcal{P} \subset \mathcal{C}$.

Definitions 2.2 — Let $\mathcal{F}$ be a family of multifunctions from X to Y. Define a mapping $w : \mathcal{F} xX \to Y$ by $w(f, x) = f(x)$ for all $(f, x) \in \mathcal{F} xX$. Then w is a multifunction. The continuity of w depends on the topology for $\mathcal{F}$. Consequently, a topology $\mathcal{J}$ for $\mathcal{F}$ is said to be jointly continuous (jointly upper semicontinuous, jointly lower semi-continuous, respectively) if w is continuous (upper semi-continuous, lower semi-continuous). We abbreviate these by j.c.; j.u.s.c.; j.l.s.c.; respectively. $\mathcal{J}$ is said to be jointly continuous (jointly upper semi-continuous, jointly lower semi-continuous respectively) on compacta if for each compact subset K of X, the restriction of w to $\mathcal{F} \times K$ is continuous (upper semi-continuous, lower semi-continuous) on $\mathcal{F} \times K$.

Theorem 2.3 — The compact open topology $\mathcal{C}$ for the family C of all continuous multifunctions from a locally compact regular space X to a space Y is the smallest jointly continuous topology on C.

PROOF: First, we shall prove that under the given conditions $\mathcal{C}$ is jointly continuous. Let U be an open subset of Y such that $w(f, x) \cap U \neq \phi$, where $f \in C$ and $x \in X$. Therefore $f(x) \cap U \neq \phi$. That is, $x \in f^-(U)$. Since X is locally compact and regular and f is l.s.c., there exists a neighbourhood $V(x)$ of x such that $x \in V(x) \subset \overline{V(x)} \subset f^-(U)$ and $\overline{V(x)}$ is compact. Let $K = \overline{V(x)}$. Since $K \subset f^-(U)$, $f(x) \cap U \neq \phi$ for all $x \in K$. This implies that $f \in W_1(K, U)$. Also, if $g \in W_1(K, U)$ and $y \in V(x)$, then $g(y) \cap U \neq \phi$. Therefore, $W_1(K, U) \times V(x)$ is a neighbourhood of (f, x) such that $w(g, y) \cap U \neq \phi$ for all $(g, y) \in W_1(K, U) \times V(x)$. Hence w is lower semi-continuous. Similarly, to prove that w is upper semi-continuous, let U be an open subset of Y containing $w(f, x)$. Then $f(x) \subset U$. That is, $x \in f^+(U)$. Since f is u.s.c. and X is locally compact and regular, there exists a neighbourhood $W(x)$ of x such that $x \in W(x) \subset \overline{W(x)} \subset f^+(U)$ and $\overline{W(x)}$ is compact. Then $W_2(\overline{W(x)}, U)$ is a neighbourhood of f and $W(x)$ is a neighbourhood of x such that $g(y) \subset U$ for all $g \in W_2(W(x), U)$ and $y \in W(x)$. This proves that w is u.s.c. Hence $\mathcal{C}$ is jointly continuous.

Now, let $\mathcal{J}$ be any other jointly continuous topology on C. We shall prove that $\mathcal{C} \subset \mathcal{J}$. Let $W_1(K, U)$ be any $\mathcal{C}$-subbasic open set and let $f \in W_1(K, U)$. Then $f(x) \cap U \neq \phi$ for all $x \in K$. Since $\mathcal{J}$ is jointly lower semi-continuous, there exists a $\mathcal{J}$-neighbourhood $W(f, x)$ of f and an open neighbourhood V_x of x such that $g(y) \cap U \neq \phi$ for all $g \in W(f, x)$ and all $y \in V_x$. Since K is compact, the open covering $\{V_x : x \in K\}$ of K admits of a finite subcovering, say $V_{x_1}, \ldots, V_{x_n}$. Let $W_f = \bigcap_{i=1}^{n} W(f, x_i)$. Then W_f is a neighbourhood of f such that $f \in W_f \subset W_1(K, U)$, for if $g \in W_f$ and $x \in K$, then $x \in V_{x_i}$ for some i, and therefore $g(x) \cap U \neq \phi$. Hence $W_1(K, U) \in \mathcal{J}$. Similarly, we can prove that every $\mathcal{C}$-subbasic open set $W_2(K, U)$ containing f is a neighbourhood of f in $\mathcal{J}$. Hence $\mathcal{C} \subset \mathcal{J}$.

Theorem 2.4 — Let $\mathcal{F}$ be the family of all point-compact continuous multifunctions from X to a Hausdorff space Y. Then a compact jointly continuous topology $\mathcal{J}$ on $\mathcal{F}$ is identical with the compact open topology $\mathcal{C}$ on $\mathcal{F}$.

PROOF : From the proof of the second part of Theorem 2.3, it follows that $\mathcal{C} \subset \mathcal{J}$. Also, if Y is Hausdorff and each $f \in \mathcal{F}$ is point compact, the point-open and hence the compact-open topology is Hausdorff. Therefore, the identity map from $(\mathcal{F}, \mathcal{J})$ to $(\mathcal{F}, \mathcal{C})$ is a continuous map from a compact space to a Hausdorff space. Hence $\mathcal{J} \equiv \mathcal{C}$.

Lemma 2.5 — If f is an upper semi-continuous function from X to Y and U is a subset of X and V is a subset of Y such that $f(x) \cap V \neq \phi$ for all $x \in U$. Then $f(x_0) \cap \bar{V} \neq \phi$ for all $x_0 \in \bar{U}$.

PROOF : If possible, let there exist $x_0 \in \bar{U}$ such that $f(x_0) \cap \bar{V} = \phi$. Then $f(x_0) \subset (Y \sim \bar{V})$. Since f is upper semi-continuous, $f^{+}(Y \sim \bar{V})$ is an open subset of X containing x_0. Since $x_0 \in \bar{U}$, therefore, $f^{+}(Y \sim \bar{V}) \cap U \neq \phi$. Let $x \in f^{+}(Y \sim \bar{V}) \cap U$. Then $f(x) \cap \bar{V} = \phi$ and therefore $f(x) \cap V = \phi$ which is a contradiction. Hence $f(x_0) \cap \bar{V} \neq \phi$ for all $x_0 \in \bar{U}$.

Lemma 2.6 : (Smithson 1973) — Let $f : X \to Y$ be a lower semi-continuous multifunction on the space X into the regular space Y. If $K \subset X$ is compact, and if $f(x) \cap V \neq \phi$ for all $x \in K$, where V is an open subset of Y, there exists an open set $W \subset Y$ such that $f(x) \cap W \neq \phi$ for all $x \in K$ and $\bar{W} \subset V$.

Theorem 2.7 — If X is locally compact, Hausdorff and Y is regular, then the compact open topology $\mathcal{C}$ for the family $\mathcal{F}$ of all point-compact continuous multifunctions on X to Y is jointly continuous.

PROOF : We have to prove that the mapping $w : \mathcal{F} \times X \to Y$ is continuous. To prove that the mapping w is lower semi-continuous, let $(f, x) \in \mathcal{F} \times X$ and U be an open subset of Y such that $w(f, x) \cap U \neq \phi$. Then $f(x) \cap U \neq \phi$. From Lemma 2.6 there exists an open subset V of Y such that $f(x) \cap V \neq \phi$ and $\bar{V} \subset U$. Since f is lower semi-continuous and X is locally compact and Hausdorff there exists a neighbourhood N_x of x such that $f(y) \cap \bar{V} \neq \phi$ for all $y \in N_x$ and $\bar{N}_x$ is compact. From Lemma 2.5, $f(y) \cap \bar{V} \neq \phi$ for all $y \in \bar{N}_x$. Since $\bar{V} \subset U, f(y) \cap U \neq \phi$ for all $y \in \bar{N}_x$. Thus $W_1(\bar{N}_x, U)$ is a neighbourhood of f. If $g \in W_1(\bar{N}_x, U)$ and $z \in N_x$, then $g(z) \cap U \neq \phi$ and hence w is lower semi-continuous.

To prove that w is upper semi-continuous, let U be an open subset of Y such that $f(x) = w(f, x) \subset U$. Since $f(x)$ is compact and Y is regular, there exists an open subset V of Y such that $f(x) \subset V \subset \bar{V} \subset U$. Also, let N_x be a neighbourhood of x with compact closure such that $f(N_x) \subset V$. Then $W_2(\bar{N}_x, U)$ and N_x are neighbourhoods of f and x, respectively such that for all $g \in W_2(\bar{N}_x, U)$ and all $y \in N_x, g(y) \subset U$. Thus w is upper semi-continuous. Hence w is continuous and $\mathcal{C}$ is jointly continuous.

Let $\mathcal{F}$ be a family of multifunctions from a topological space $(X, \mathcal{J})$ to a uniform space $(Y, \mathcal{V})$. If K is any subset of X and $V \in \mathcal{V}$, let $W(K/V) = \{(f, g) \in \mathcal{F} \times \mathcal{F}$: for each $x \in K$, $(f(x) \times \{y\}) \cap V \neq \phi$ for all $y \in g(x)$ and $(\{z\} \times g(x)) \cap V \neq \phi$ for all $z \in f(x)\}$.

Definition 2.8 — A net $\{f_\nu : \nu \in (D, >)\}$ in $\mathcal{F}$ is said to converge uniformly to $f \in \mathcal{F}$ on a subset K of X if for every $V \in \mathcal{V}$ there exists an index ν' such that for all $\nu > \nu'$, $(f, f_\nu) \in W(K/V)$. A net $\{f_\nu : \nu \in (D, >)\}$ in $\mathcal{F}$ converges uniformly on compacta if it converges uniformly on each compact subset of X.

Theorem 2.9 — Let C be the class of all point-compact continuous multifunctions from a topological space X to a uniform space $(Y, \mathcal{V})$. Let $\mathcal{C}$ be the compact open topology on C. Then a net $\{f_\nu : \nu \in (D, >)\}$ in $(C, \mathcal{C})$ converges if and only if it converges uniformly on compacta.

PROOF : Let $\{f_\nu : \nu \in (D, >)\}$ converge uniformly on compacta to f and let $f \in W_1(K, U)$, where K is a compact subset of X and U is an open subset of Y. Let $y_x \in f(x) \cap U$ for each $x \in K$. There is a $V_x \in \mathcal{V}$ such that $y_x \in V_x[y_x] \subset U$. Let $W_x \in \mathcal{V}$ such that $W_x \circ W_x \subset V_x$. Then

$$y_x \in (W_x[y_x])^0 \subset W_x[y_x] \subset W_x \circ W_x[y_x] \subset V_x[y_x] \subset U.$$

Therefore $\{f^-((W_x[y_x])^0) : x \in K\}$ is an open covering of K. Since K is compact, this covering admits of a finite sub-covering, say $f^-((W_{x_1}[y_{x_1}])^0), f^-((W_{x_2}[y_{x_2}])^0, \ldots, f^-((W_{x_n}[y_{x_n}])^0)$. Since $\{f_\nu : \nu \in (D >)\}$ converges uniformly on compacta to f, for each $i = 1, 2, \ldots, n$ there exists a $\nu_i \in D$ such that for all $\nu > \nu_i$, $(f, f_\nu) \in W(K/W_{x_i})$. Let $\nu' = \max\{\nu_1, \nu_2, \ldots, \nu_n\}$. Let $\nu > \nu'$. Then $\nu > \nu_i$ for all i and hence

$$(f, f_\nu) \in W(K/W_{x_i}) \text{ for all } i.$$

We shall prove that $f_\nu \in W_1(K, U)$. Let $x \in K$. Then $x \in f^-((W_{xi}[y_{x_i}])^0)$ for some $i \in \{1, 2, \ldots, n\}$. Let $z_x \in f(x) \cap (w_{x_i}[y_{x_i}])^0$. Then $(\{z_x\} \times f_\nu(x)) \cap W_{x_i} \neq \phi$. If $z_\nu \in f_\nu(x)$ such that $(z_x, z_\nu) \in W_{x_i}$, then $z_\nu \in W_{x_i}[z_x] \subset W_{x_i} \circ W_{x_i}[y_x] \subset U$. Hence $f_\nu(x) \cap U \neq \phi$. This proves that $f_\nu \in W_1(K, U)$.

Now, if $f \in W_2(K, U)$, then $f(K) \subset U$. Since $f(K)$ is compact, there exists $V \in \mathcal{V}$ such that $V[f(K)] \subset U$. Since $\{f_\nu : \nu \in (D, >)\}$ converges uniformly on K to f, there exists $\nu' \in D$ such that for all $\nu > \nu'$, $(f, f_\nu) \in W(K/V)$. Let $x \in K$ and $\nu > \nu'$. Then $(f(x) \times \{z\}) \cap V \neq \phi$ for all $z \in f_\nu(x)$. That is, for each $z \in f_\nu(x)$, there is a $z_x \in f(x)$ such that $(z_x, z) \in V$. Therefore $z \in V[z_x] \subset V[f(K)] \subset U$ for each $z \in f_\nu(x)$. Hence $f_\nu(x) \subset U$. Therefore $f_\nu \in W_2(K, U)$ for all $\nu > \nu'$. Hence $\{f_\nu : \nu \in (D, >)\}$ converges to f.

Conversely, let $\{f_\nu : \nu \in (D, >)\}$ converge to f and let K be a compact subset of X. Let $V \in \mathcal{V}$ and $x \in K$. Let U be a symmetric member of $\mathcal{V}$ such that

$U \circ U \circ U \circ U \circ U \circ U \subset V$. For each $y \in f(x)$, $(U[y])^0$ is an open set containing y. Since $f(x)$ is compact, there exists a finite number of points $y_1, y_2, \ldots, y_{n(x)}$ such that $f(x) \subset \bigcup_{i=1}^{n(x)} (U[y_i])^0$. Also $f(x) \cap (U[y_i])^0 \neq \phi$ for each $i = 1, 2, \ldots, n(x)$. Therefore $x \in f^+(\bigcup_{i=1}^{n(x)} (U[y_i])^0) \cap (\bigcap_{i=1}^{n(x)} f^-((U[y_i])^0))$. Since f is continuous, it follows that $f^+(\bigcup_{i=1}^{n(x)} (U[y_i])^0) \cap (\bigcap_{i=1}^{n(x)} f^-((U[y_i])^0))$ ($= G$, say) is an open neighbourhood of x. Since K is compact, there exists finitely many points $x_1, x_2, \ldots, x_k$ of K such that $K \subset G_1 \cup G_2 \cup \ldots \cup G_k$ where $x_j \in G_j \subset f^+(\bigcup_{i=1}^{n(j)} (U[y_i])^0) \cap (\bigcap_{i=1}^{n(j)} f^-(U[y_i])^0)$. For each $j = 1, 2, \ldots, k$, let $K_j = K \cap \bar{G}_j$.

Each K_j is compact. Now

$$K_j \subset \bar{G}_j \subset [\bigcap_{i=1}^{n(j)} f^-((U[y_i])^0)]^-$$

$$\subset \bigcap_{i=1}^{n(j)} f^-(\overline{(U[y_i])^0}) \subset \bigcap_{i=1}^{n(j)} f^-(U[(U[y_i])^0])$$

$$\subset \bigcap_{i=1}^{n(j)} f^-((U \circ U \circ U[y_i])^0)$$

for $x' \in U[(U[y_i])^0] \Rightarrow (x', x'') \in U$ for some $x'' \in (U[y_i])^0 \Rightarrow x' \in U[x'']$, and there exists $W \in \mathcal{V}$ such that $x'' \in W[x''] \subset U[y_i] \Rightarrow x' \in U[x'] \subset U \circ U[x''] \subset U \circ U \circ W[x''] \subset U \circ U \circ U[y_i]$.

Therefore, $f(t) \cap (U \circ U \circ U[y_i])^0 \neq \phi$ for each $t \in K_j$ and each $y_i \in f(x_j)$ for $i = 1, 2, \ldots, n(j)$. That is $f \in \bigcap_{i=1}^{n(j)} W_1(K_j, (U \circ U \circ U[y_i])^0)$. Hence

$$f \in \bigcap_{j=1}^{k} \bigcap_{i=1}^{n(j)} W_1(K_j, (U \circ U \circ U[y_i])^0).$$

Also,

$$K_j \subset \bar{G}_j \subset [f^+(\bigcup_{i=1}^{n(j)} (U[y_i])^0)]^- \subset f^+[(\bigcup_{i=1}^{n(j)} (U[y_i])^0)^-]$$

$$= f^+[\bigcup_{i=1}^{n(j)} (U[y_i])^{0-}]$$

$$\subset f^+[\bigcup_{i=1}^{n(j)} (U \circ U \circ U[y_i])^0].$$

Therefore, $f(K_j) \subset \bigcup_{i=1}^{n(j)} (U \circ U \circ U[y_i])^0$ for all $j = 1, 2, \ldots, k$.

That is, $f \in \bigcap_{j=1}^{k} W_2(K_j, \bigcup_{i=1}^{n(j)} (U \circ U \circ U[y_i])^0)$.

Hence $f \in [\bigcap_{j=1}^{k} \bigcap_{i=1}^{n(j)} W_2(K_j,(U \circ U \circ U[y_i])^0) \cap [\bigcap_{j=1}^{k} W_2(K_j, \bigcup_{i=1}^{n(j)} (U \circ U \circ U[y_i])^0)$ $(= W$ say). W is an open subset of $\mathcal{F}$ in the compact open topology. Since $\{f_\nu : \nu \in (D, >)\}$ converges to f, therefore there exists an index ν' such that $f_\nu \in W$ for all $\nu > \nu'$. Now let $x \in K$. Then $x \in K_j$ for some j. Therefore, for each $\nu > \nu', f_\nu(x) \cap (U \circ U \circ U[y_i])^0 \neq \phi$ for each $i = 1, 2, \ldots, n(j)$ and

$$f_\nu(x) \subset \bigcup_{i=1}^{n(j)} (U \circ U \circ U[y_i])^0.$$

Also $f \in W$ implies that $f(x) \cap (U \circ U \circ U[y_i])^0 \neq \phi$ for each $i = 1, 2, \ldots, n(j)$ and $f(x) \subset \bigcup_{i=1}^{n(j)} (U \circ U \circ U[y_i])^0$. Let $y \in f(x)$. Then $y \in (U \circ U \circ U[y_i])^0$ for some $y_i \in f(x_j)$. Therefore $(y, y_i) \in U \circ U \circ U$. Also for that y_i, $f_\nu(x) \cap (U \circ U \circ U[y_i]) \neq \phi$ which implies that there is some $z \in f_\nu(x)$ such that $z \in U \circ U \circ U[y_i]$, that is $(z, y_i) \in U \circ U \circ U$. Since U is symmetric, therefore $(y, y_i) \in U \circ U \circ U$ and $(z, y_i) \in U \circ U \circ U$ imply that $(y, z) \in U \circ U \circ U \circ U \circ U \circ U \subset V$ and hence $(\{y\} \times f_\nu(x)) \cap V \neq \phi$ for all $y \in f(x)$. Similarly, $(f(x) \times \{z\}) \cap V \neq \phi$ for all $z \in f_\nu(x)$. Thus $(f, f_\nu) \in W(K/V)$ for all $\nu > \nu'$, which proves that f_ν converges on compacts to f.

3. Graph Topologies

Definition 3.1 : (Poppe 1967 and Naimpally and Pareek 1970) — A topology on Y^X generated by a sub-basis consisting of sets of the form $(A, U) = \{f \in Y^X : f(A) \subset U\}$, where A is a closed subset of X and U is an open subset of Y, is called the graph topology Γ_1.

Definitions 3.2 : (Naimpally and Pareek 1970) — (i) The topology on Y^X generated by a sub-basis consisting of sets of the form $F_U = \{f \in Y^X : G(f) \subset U\}$, where U is an open subset of $X \times Y$, is called the graph topology Γ_2.

(ii) The topology on Y^X generated by a sub-basis consisting of all sets of the form $\{f \in Y : G(f) \subset \bigcup_{i \in I} U$ and $G(f) \cap U_i \neq \phi$ for all $i \in I\}$, where U_i's are open subsets of $X \times Y$ and I is a finite set, is called the graph topology Γ_3.

(iii) The topology on Y^X generated by a sub-basis consisting of all sets of the form $\{f \in Y^X : G(f) \subset \bigcup_{i \in I} U_i$ and $G(f) \cap U_i \neq \phi$ for all $i \in I\}$, where $\{U_i : i \in I\}$ is a locally finite family of open subsets of $X \times Y$, is called the graph topology Γ_4.

The topology Γ_1 was defined and studied first by Poppe (1967) and the topology Γ_2 was first introduced by Naimpally (1966).

We now generalize the above four graph topologies to include the family of multifunctions Y^{mX} as follows.

Definitions 3.3 — (i) Let Γ_{m_1} be the topology for Y^{mX} generated by a sub-basis consisting of all sets $(A, U)_1$ and $(A, U)_2$, where

$$(A, U)_1 = \{f \in Y^{mX} : f(x) \cap U \neq \phi \text{ for all } x \in A\} \text{ and}$$

$$(A, U)_2 = \{f \in Y^{mX} : f(A) \subset U\},$$

where A is a closed subset of X and U is an open subset of Y.

(ii) Let Γ_{m_2} be the topology for Y^{mX} generated by a sub-basis consisting of all sets F_U^- and F_U^+ where $F_U^- = \{f \in Y^{mX} : (\{x\} \times f(x)) \cap U \neq \phi$ for all $x \in X\}$ and $F_U^+ = \{f \in Y^{mX} : G(f) \subset U\}$, U being an open subset of $X \times Y$.

(iii) Let Γ_{m_3} be the topology for Y^{mX} generated by a sub-basis consisting of all sets of the form $\{f \in Y^{mX}$: for each $x \in X$, $(\{x\} \times f(x)) \cap (\bigcup_{i=1}^{n} U_i) \neq \phi$ and for each $i = 1, 2, \ldots, n$ there exists $x_i \in X$ such that $(\{x_i\} \times f(x_i)) \cap U_i \neq \phi\}$ and $\{f \in Y^{mX}$: $G(f) \subset \bigcup_{i=1}^{n} U_i$ and for each $i = 1, 2, \ldots, n$ there exists $x_i \in X$ such that $(\{x_i\} \times f(x_i)) \subset U_i\}$ where U_i's are open subsets of $X \times Y$.

(iv) The topology Γ_{m_4} is the same as Γ_{m_3} except that a finite family $\{U_i, i = 1, 2, \ldots, n\}$ of open subsets of $X \times Y$ is replaced by an arbitrary locally finite family of open subsets of $X \times Y$.

The above Definitions 3.3 of graph topologies for multifunctions are reduced to the Definitions 3.1 and 3.2 for single valued mappings, if the family of multifunctions is replaced by the family of single valued mappings.

It can easily be seen that $\Gamma_{m_2} \subset \Gamma_{m_3} \subset \Gamma_{m_4}$. Now we shall prove that $\Gamma_{m_1} \subset \Gamma_{m_2}$.

Theorem 3.4 — Topology Γ_{m_1} is weaker than the topology Γ_{m_2}.

PROOF: Let $(A, U)_1$ be any Γ_{m_1} sub-basic open set. Let $f \in (A, U)_1$ so that $f(x) \cap U \neq \phi$ for all $x \in A$. Let $V = (X \sim A) \times Y \cup (X \times U)$, then V is an open subset of $X \times Y$. Consider the set $F_V^- = \{h \in Y^{mX} : (\{x\} \times h(x)) \cap V \neq \phi$ for all $x \in X\}$. Clearly $f \in F_V^- \subset (A, U)_1$. Hence $(A, U)_1$ is Γ_{m_2}-open. Similarly, if

$(A, U)_2$ is another sub-basic open set containing $f \in Y^{mX}$ then $f \in F_W^+ \subset (A, U)_2$, where $W = (X \sim A) \times Y \cup (X \times U)$. Hence $\Gamma_{m1} \subset \Gamma_{m2}$.

Thus $\Gamma_{mi} \subset \Gamma_{mi+1}$ for $i = 1, 2, 3$. Poppe (1967) has constructed an example to show that $\Gamma_{m_1} \neq \Gamma_{m_2}$, in general. Naimpally and Pareek (1970) have constructed examples showing that $\Gamma_{m_2}, \Gamma_{m_3}, \Gamma_{m_4}$ are all distinct.

The following theorems give conditions under which some of these topologies are equivalent.

Theorem 3.5 — If X is a compact Hausdorff space, then $\Gamma_{m_1} = \Gamma_{m_2}$ on the family C of all continuous point-compact multifunctions from X to Y.

PROOF: We have to prove that under the given conditions $\Gamma_{m_2} \subset \Gamma_{m_1}$. Let F_U^- be a Γ_{m_2} sub-basic open set containing f so that for all $x \in X$, $(\{x\} \times f(x)) \cap U \neq \phi$ where U is an open subset of $X \times Y$. Let $U = \bigcup_{\alpha \in \Delta} (U_\alpha \times V_\alpha)$ where each U_α is open in X and each V_α is open in Y. Since $(\{x\} \times f(x)) \cap U \neq \phi$ for all $x \in X$, there exists some $\alpha \in \Lambda$ such that $(\{x\} \times f(x)) \cap (U_\alpha \times V_\alpha) \neq \phi$. Thus $x \in U_\alpha$ and $f(x) \cap V_\alpha \neq \phi$ so that $x \in U_\alpha \cap f^-(V_\alpha)$. Since f is continuous, $f^-(V_\alpha)$ is open in X. Since X is regular, there exists an open subset U_x of X such that $x \in U_x \subset \bar{U}_x \subset U_\alpha \cap f^-(V_\alpha)$. Since X is compact, there exist finitely many points $x_1, x_2, \ldots, x_n$ of X and the corresponding sets $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ such that $X \subset \bigcup_{i=1}^{n} U_{x_i}$.

Let U_{α_i} and V_{α_i} be the sets such that $x_i \in U_{x_i} \subset \bar{U}_{x_i} \subset U_{\alpha_i} \cap f^-(V_{\alpha_i})$. Now $\bar{U}_{x_i} \subset f^-(V_{\alpha_i})$ implies that $f(t) \cap V_{\alpha_i} \neq \phi$ for all $t \in \bar{U}_{x_i}$. Therefore $f \in \bigcap_{i=1}^{n} (\bar{U}_{x_i}, V_{\alpha_i})_1$.

Let $g \in \bigcap_{i=1}^{n} (\bar{U}_{x_i}, V_{\alpha_i})_1$, and let $x \in X$. Then $x \in U_{x_i}$ for some i and therefore $g \in (\bar{U}_{xi}, V_{\alpha_i})_1$ implies that $g(x) \cap V_{\alpha_i} \neq \phi$. Also $U_{x_i} \subset U_{\alpha_i}$ implies that $x \in U_{\alpha_i}$. Therefore, $(\{x\} \times g(x)) \cap (U_{\alpha_i} \times V_{\alpha_i}) \neq \phi$. That is $(\{x\} \times g(x)) \cap U \neq \phi$. Thus $f \in \bigcap_{i=1}^{n} (\bar{U}_{x_i}, V_{\alpha_i})_1 \subset F_U^-$. Thus every F_U^- is open in Γ_{m_1}.

Again, let F_U^+ be another Γ_{m_2} subbasic open set containing f. Then $G(f) \subset U$ and U is an open subset of $X \times Y$. Let $U = \bigcup_{\alpha \in \Delta} (U_\alpha \times V_\alpha)$, where each U_α is an open subset of X and each V_α is an open subset of Y. For each $x \in X$, $(\{x\} \times f(x))$

$\subset \bigcup_{\alpha \in \Lambda} (U_\alpha \times V_\alpha)$. Since $f(x)$ is compact, therefore for each $x \in X$ there exists a finite subset $\Lambda(x)$ of Λ such that $x \in \bigcap_{\alpha \in \Lambda(x)} U_\alpha$ and $f(x) \subset \bigcup_{\alpha \in \Lambda(x)} V_\alpha$ that is, $x \in (\bigcap_{\alpha \in \Lambda(x)} U_\alpha) \cap f^+(\bigcup_{\alpha \in \Lambda(x)} V_\alpha)$. Let U_x be an open subset of X such that $x \in U_x \subset \bar{U}_x \subset (\bigcap_{\alpha \in \Lambda(x)} U_\alpha) \cap f^+(\bigcup_{\alpha \in \Lambda(x)} V_\alpha)$. Since X is compact there exist finitely many points $x_1, x_2, \ldots, x_n$ of X such that $X = \bigcup_{i=1}^{n} U_{x_i}$ and $x_i \in U_x \subset \bar{U}_{x_i} \subset (\bigcap_{\alpha \in \Lambda(x_i)} U_\alpha) \cap f^+(\bigcup_{\alpha \in \Lambda(x_i)} V_\alpha)$. Since $\bar{U}_{x_i} \subset f^+(\bigcup_{\alpha \in \Lambda(x_i)} V_\alpha)$, therefore $f \in \bigcap_{i=1}^{n} (\bar{U}_{x_i}, \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha)_2$. Now, let $g \in \bigcap_{i=1}^{n} (\bar{U}_{x_i}, \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha)_2$ and let $x \in X$. Then $x \in U_{x_i}$ for some i. Therefore $g \in (\bar{U}_{x_i}, \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha)_2$, implies that $g(x) \subset \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha$. Also $x \in U_{x_i} \subset \bigcap_{\alpha \in \Lambda(x_i)} U_\alpha$ implies that $x \in U_\alpha$ for all $\alpha \in \Lambda(x_i)$. Therefore $\{x\} \times g(x) \subset \bigcup_{\alpha \in \Lambda(x_i)} (U_\alpha \times V_\alpha) \subset U$. Hence $G(g) \subset U$. Thus $f \in \bigcap_{i=1}^{n} (\bar{U}_{x_i}, \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha)_2 \subset F_U^+$ which implies that F_U^+ is open in Γ_{m_1}. Hence $\Gamma_{m_2} \subset \Gamma_{m_1}$.

Theorem 3.6 — A space X is T_1 if and only if the two topologies Γ_{m_2} and Γ_{m_3} are identical on Y^{mX} for all Y.

PROOF : If X is not a T_1-space, then there exists a pair a, b of distinct points of X such that every neighbourhood of a contains the point b. Let $Y = \{c, d\}$ with the discrete topology. Let $f, g \in Y^{mX}$ be defined as

$$\begin{aligned} f(x) &= c, \text{ if } x \neq b \\ &= d, \text{ if } x = b \\ g(x) &= c, \text{ for all } x \in X. \end{aligned}$$

Let $U_1 = U_\alpha \times \{d\}$ be an open subset of $X \times Y$ containing (b, d), where U_α is an open subset of X containing b. Let $U_2 = \{(x, c) : x \in X\}$. Then $G(f) \subseteq U_1 \cup U_2$ and for each $i = 1, 2$, there exists an $x_i \in X$ such that $\{x_i\} \times f(x_i) \subset U_i$. Thus $O_2 = \{h \in Y^{mX} : G(h) \subset U_1 \cup U_2$ and for each $i = 1, 2$, there exists $x_i \in X$ such that $\{x_i\} \times h(x_i) \subset U_i\}$ is a Γ_{m_3}-open set containing f. We shall show that no Γ_{m_2} basic neighbourhood of f is contained in O_2. For if, $\left(F^+_{U'_1} \cap \ldots \cap F^+_{U'_n}\right) \cap \left(F^-_{V'_1} \cap \ldots \cap F^-_{V'_m}\right)$ be any Γ_{m_2}-basic neighbourhood of f, then $f \in \bigcap_{i=1}^{n} F^+_{U'_i}$ implies that

$G(f) \subset \bigcap_{i=1}^{n} U'_i$. Since every neighbourhood of a contains b, therefore $G(g) \subset \bigcap_{i=1}^{n} U'_i$. Thus $g \in \bigcap_{i=1}^{n} F^{+}_{U'_i}$. But $g \notin O_2$, since $G(g) \cap U_1 = \phi$. Hence $\bigcap_{i=1}^{n} F^{+}_{U'_i}$ is not contained in O_2. Similarly, $\bigcap_{i=1}^{m} F^{-}_{V'_i} \not\subset O_2$. Hence O_2 is a Γ_{m_3} open set which is not Γ_{m_2}-open. Thus the condition is necessary.

Conversely, let $O_1 = \{h \in Y^{mX}$: for each $x \in X$, $(\{x\} \times h(x)) \cap (\bigcup_{i=1}^{n} U_i) \neq \phi$ and for each $i = 1, 2, \ldots, n$ there exists $x_i \in X$ such that $(\{x_i\}) \times h(x_i)) \cap U_i \neq \phi\}$, where U_i's are open subsets of $X \times Y$, be any Γ_{m_3}-sub-basic open set containing $f \in Y^{mX}$. For each $i = 1, 2, \ldots, n$ let $p_i \in X$ such that $p_i \times f(p_i) \cap U_i \neq \phi$. Let $q_i \in f(p_i)$ such that $(p_i, q_i) \in U_i$. Since U_i is open and X is T_1, there exists a W_{p_i}, open in $X \times Y$, such that $(p_i, q_i) \in W_{p_i} \subset U_i \sim (P \sim \{p_i\} \times Y)$, where $P = \{p_1, p_2, \ldots, p_n\}$. Thus W_{p_i} is such that $(\{p_i\} \times f(p_i)) \cap W_{p_i} \neq \phi$ and $(\{p_i\} \times f(p_i)) \cap W_{p_j} = \phi$ for all $j \neq i$. For each $x \in X \sim P$, let $y_x \in f(x)$ be such that $(x, y_x) \in (\bigcup_{i=1}^{n} U_i) \sim (P \times Y)$. For $x \in X \sim P$ let W_x be an open subset of $X \times Y$ such that $(x, y_x) \in W_x \subset (\bigcup_{i=1}^{n} U_i) \sim P \times Y$. Let $W = \bigcup_{x \in X} W_x$. Then W is an open subset of $X \times Y$ such that $f \in F^{-}_{W}$. Let $g \in F^{-}_{W}$. Then for each $x \in X$, $(\{x\} \times g(x)) \cap W \neq \phi$ implies that $(\{x\} \times g(x)) \cap (\bigcup_{i=1}^{n} U_i) \neq \phi$ for all $x \in X$. In particular for each $i = 1, 2, \ldots, n$ $(\{p_i\} \times g(p_i)) \cap W \neq \phi$ implies that $(\{p_i\} \times g(p_i)) \cap (\bigcup_{j=1}^{n} W_{p_j}) \neq \phi$. Since $(\{p_i\} \times Y) \cap W_{p_j} = \phi$ for all $j \neq i$, therefore $(p_i \times g(p_i)) \cap W_{p_i} \neq \phi$. Hence $(\{p_i\} \times g(p_i)) \cap U_i \neq \phi$. Thus $g \in O_1$ and O_1 is Γ_{m_2}-open. Similarly, let $O_2 = \{h \in Y^{mX} : G(h) \subset \bigcup_{i=1}^{n} U_i$ and for each i there exists a $p_i \in X$ such that $\{p_i\} \times f(p_i) \subset U_i\}$ be Γ_{m_3}-subbasic open set containing f. Then $G(f) \subset \bigcup_{i=1}^{n} U_i$ and for each i, let $p_i \in X$ such that $\{p_i\} \times f(p_i) \subset U_i$. Since X is T_1, for each p_i, there exists an open subset W_{p_i} of $X \times Y$ such that $\{p_i\} \times f(p_i) \subset W_{p_i} \subset U_i \sim ((P \sim \{p_i\}) \times Y)$, where $P = \{p_1, p_2, \ldots, p_n\}$. For each $x \in X \sim P$, $\{x\} \times f(x) \subset (\bigcup_{i=1}^{n} U_i) \sim (P \times Y)$. Let W_x be an open subset of $X \times Y$ such that

$$\{x\} \times f(x) \subset W_x \subset (\bigcup_{i=1}^{n} U_i) \sim (P \times Y).$$

Let $W = \bigcup_{x \in X} W_x$. Then F_W^+ is a Γ_{m_2}-open set containing f. We shall prove that $F_W^+ \subset O_2$. Let $g \in F_W^+$, then obviously $G(g) \subset \bigcup_{i=1}^{n} U_i$. Also for each $i = 1, 2, \ldots, n$, $\{p_i\} \times g(p_i) \subset \bigcup_{j=1}^{n} W_{p_j}$. By construction of W_p's, $W_{p_j} \cap (\{p_i\} \times g(p_i)) = \phi$ for all $i \neq j$. Therefore $\{p_i\} \times g(p_i) \subset W_{p_i} \subset U_i$. Hence $g \in O_2$, which proves that O_2 is a Γ_{m_2}-open subset of Y^{mX}. Hence $\Gamma_{m_3} \subset \Gamma_{m_2}$. This proves the theorem.

Theorem 3.7 — If X is a T_1 space, then the pointwise convergence topology $\mathcal{P}$ for Y^{mX} is contained in Γ_{m_i} $(i = 1, 2, 3, 4)$.

PROOF : It is sufficient to prove that $\mathcal{P} \subset \Gamma_{m_1}$. Since $\Pi_x^-(U) = (\{x\}, U)_1$ and $\Pi_x^+(U) = (\{x\}, U)_2$ and X is T_1, the result follows.

Theorem 3.8 — Let X be a T_1 space and let Y be a non-degenerate space (that is, there exists an open set V in Y such that $\phi \neq V \neq Y$). Then the pointwise convergence topology $\mathcal{P}$ on Y^{mX} equals Γ_{m_1} if and only if X is a finite set with the discrete topology.

PROOF : The condition is sufficient. Let X be a finite discrete space. Let $(A, U)_1$ be any Γ_{m_1}-sub-basic open set containing $f \in Y^{mX}$, so that $f(x) \cap U \neq \phi$ for all $x \in A$. Let $A = \{x_i : i = 1, 2, \ldots, n\}$. Then $\bigcap_{i=1}^{n} \Pi_{x_i}^-(U)$ is a $\mathcal{P}$-open set containing f and contained in $(A, U)_1$. Thus $(A, U)_1$ is $\mathcal{P}$-open. Similarly, $(A, U)_2$ is $\mathcal{P}$-open. Thus $\mathcal{P} \supset \Gamma_{m_1}$. Hence from Theorem 3.7, $\mathcal{P} = \Gamma_{m_1}$.

The condition is necessary. We only need to prove that X is finite, because every finite T_1 space is discrete. Let X be non-finite and let X have the cofinite topology. Let V be a proper open subset of Y. Let $y \in V$. Then the constant mapping f which maps X onto $\{y\}$ is in $(X, V)_2$, but for every basic set $\bigcap_{i=1}^{n} \Pi_{x_i}^-(U)$ in the topology of pointwise convergence we can find a function g such that $g(x_i) \cap U_i \neq \phi$, $i = 1, 2, \ldots, n$ and $g(x) \subset Y \sim V$ for some $x \neq x_i$, $i = 1, 2, \ldots, n$. Hence $\Gamma_{m_1} \not\subset \mathcal{P}$ which is a contradiction. Hence X must be finite.

Theorem 3.9 — If X is Hausdorff, the compact open topology is contained in the Γ_{m_1} topology and the two are equal if X is compact.

PROOF : The proof is obvious, since in a Hausdorff space every compact subset is closed and in a compact space every closed subset is compact.

As shown by Poppe there exists a non-Hausdorff space X such that $\Gamma_{m_1} \neq$ compact open topology even on the family of continuous multifunctions.

Definition 3.10 — A multifunction $f \in Y^{mX}$ is said to be Γ_i-almost continuous ($i = 1, 2, 3, 4$) if every Γ_{m_i}-neighbourhood of f contains a continuous multifunction. The set of all Γ_i-almost continuous multifunctions will be denoted by A_{m_i}.

Theorem 3.11 — The set A_{m_i} is a closed subset of (Y^{mX}, Γ_{m_i}). In fact, A_{m_i} is the Γ_{mi}-closure of the set C of all continuous multifunctions in Y^{mX}.

PROOF : Follows immediately from the definition of Γ_i-almost continuity.

Since $\Gamma_{m_i} \subset \Gamma_{m_{i+1}}$ ($i = 1, 2, 3$), it follows that a Γ_{i+1}-almost continuous multifunction is Γ_i-almost continuous. The converse, however, is not true as is shown by Naimpally and Pareek (1970, examples 4.4 to 4.6).

Theorem 3.12 — If $\mathcal{F}$ is the family of all point-closed multifunctions from X to a T_1-space Y having more than one point, then the conditions (i) X is T_1 and (ii) $(\mathcal{F}, \Gamma_{m_i})$ is T_1, are equivalent.

PROOF : Let (i) be satisfied. Let f and g be two distinct points of $\mathcal{F}$. Then there exists an $x \in X$ such that $f(x) \neq g(x)$. If $f(x) \not\subset g(x)$, let $y \in f(x) \sim g(x)$. Since Y is T_1 and g is point-closed, therefore $Y \sim \{y\}$ and $Y \sim g(x)$ are non-empty open subsets of Y containing $g(x)$ and y respectively. Since X is T_1, the set $(\{x\}, Y \sim g(x))_1$ is a Γ_{m_1}-open set containing f but not g, and $(\{x\}, Y - \{y\})_2$ is a Γ_{m_1}-open set containing g but not f.

Conversely, let (ii) be satisfied. If X is not T_1, there exist two distinct points a and b in X such that every neighbourhood of a contains b. Thus $a \in \overline{\{b\}}$ and $\{b\}$ is not a closed set. Let f, g be two functions in Y^{mX} defined as $f(x) = p$ for all $x \neq b$, $f(b) = q$ and $g(x) = p$ for all $x \in X$, where p, q are two distinct points of Y. Now consider any Γ_{m_1}-neighbourhood $(A, U)_2$ of f, where U is open in Y and A is closed in X. Since $\{b\}$ is not a closed set, therefore either A is disjoint form $\{b\}$ or A has points of X other than b. In the first case $g(A) = f(A) \subset U$. In the second case, $g(A) = \{p\} \subset \{p, q\} = f(A) \subset U$. Thus every neighbourhood of the form $(A, U)_2$ contains g. Similarly, every neighbourhood of f of the form $(A, U)_1$ contains g. Hence $(\mathcal{F}, \Gamma_{m_1})$ is not a T_1-space, which is a contradiction. Thus X must be T_1.

The above theorem can easily be proved for the topologies Γ_{m_2}, Γ_{m_3} and Γ_{m_4}.

Theorem 3.13 — If Y has at least two points then the set $\mathcal{F}$ of all point-compact multifunctions from X to Y is Hausdorff with respect to the topologies Γ_{m_i} ($i = 1, 2, 3, 4$) if and only if X is T_1 and Y is Hausdorff.

PROOF : The theorem is proved if we prove the 'if' part for Γ_{m_1} and the 'only if' part for Γ_{m_4}.

Let X be a T_1 space and Y be a Hausdorff space. Let f, g be two distinct points of $\mathcal{F}$. Then there exists a point $a \in X$ such that $f(a) \neq g(a)$. If $f(a) \not\subset g(a)$, let $y \in f(a) \sim g(a)$. Since Y is Hausdorff, there exist nonempty disjoint open subsets U and V of Y containing y and the compact set $g(a)$ respectively. Then $(\{a\}, U)_1$ and $(\{a\}, V)_2$ are disjoint Γ_{m_1}-open sets containing f and g respectively.

Conversely, suppose that $(\mathscr{F}, \Gamma_{m_4})$ is Hausdorff. If possible, let there exists two distinct points p and q of Y such that every neighbourhood of p intersects every neighbourhood of q. Let f, g be defined as $f(x) = p$ for all $x \in X$ and $g(x) = q$ for all $x \in X$. Then, we see that every Γ_{m_4}-neighbourhood of f intersects every Γ_{m_4}-neighbourhood of g, which is a contradiction. Hence Y must be Hausdorff. The proof for X to be T_1 is the same as in Theorem 3.12.

4. σ-TOPOLOGY

The σ-topologies for the class of all single-valued mappings were introduced by Arens and Dugundji (1951). For any open cover σ of X and the collection $\{A_i : i \in I\}$ of closed subsets of X such that each A_i is contained in some member of σ, the topology on Y^X generated by the sub-basis consisting of all sets of the form $(A_i, U) = \{f \in Y^X : f(A_i) \subset U\}$, U open in Y, $i \in I$, is called the σ-topology for Y^X.

The above definition can be generalized to include the family of all multifunctions from X to Y, as follows.

Definition 4.1 — Let σ be an open cover of X and let $\{A_i : i \in I\}$ be the collection of all closed subsets of X such that each A_i is contained in some member of σ. Then the topology on Y^{mX} generated by the subbase $\{(A_i, U)_1, (A_i, U)_2 : i \in I, U$ open in $Y\}$ is called the σ-topology for Y^{mX}.

Obviously, the σ-topology is the Γ_{m_1}-topology if $X \in \sigma$, and a Γ_{m_1}-topology is the finest σ-topology. Thus we have the following :

Theorem 4.2 — Any σ-topology is contained in Γ_{m_i} $(i = 1, 2, 3, 4)$.

Examples have been given by Naimpally and Pareek (1970) to show that a σ-topology is distinct from the Γ_{m_i}-topologies.

Now, let X, Y, Z be three spaces. For each multifunction $g : Z \times X \twoheadrightarrow Y$ we define a single-valued mapping $g^* : Z \to Y^{mX}$ as $g^*(z)(x) = g(z, x)$ for all $z \in Z$ and $x \in X$. Conversely, if $g^* : Z \to Y^{mX}$ is a single-valued mapping then we can define a multifunction $g : Z \times X \twoheadrightarrow Y$ as $g(z, x) = g^*(z)(x)$. The mappings g and g^* are called associated mappings. A topology $\mathscr{J}$ for Y^{mX} is said to be proper (admissible, respectively) if for any topological space Z, the continuity of a multifunction $g : Z \times X \to Y$ ($g^* : Z \to Y^{mX}$ respectively) implies the continuity of g^* (g respectively).

Theorem 4.3 — A topology $\mathscr{J}$ for Y^{mX} is admissible if and only if it is jointly continuous.

PROOF : Let $\mathcal{G}$ be an admissible topology on Y^{mX}. Then for every space Z, for which the mapping $g^* : Z \to (Y^{mX}, \mathcal{G})$ is continuous, we must have $g : Z \times X \to Y$ also to be continuous. Let $w : (Y^{mX}, \mathcal{G}) \times X \to Y$ be the evaluation map. Then $w^* : (Y^{mX}, \mathcal{G}) \to (Y^{mX}, \mathcal{G})$ is the mapping such that

$$\begin{aligned} w^*(f)\,(x) &= w(f, x) \\ &= f(x) \text{ for all } f \in Y^{mX} \text{ and for all } x \in X. \end{aligned}$$

Thus w^* is the identity mapping which is continuous. Hence w is continuous.

Conversely, let $\mathcal{G}$ be a jointly continuous topology on Y^{mX}, so that the map $w : Y^{mX} \times X \to Y$ is continuous, let $g^* : Z \to Y^{mX}$ be any continuous single-valued mapping. Let $h : Z \times X \to Y^{mX} \times X$ be the mapping defined as

$$h(z, x) = (g^*(z), x) \text{ for all } (z, x) \in Z \times X.$$

Since g^* is continuous, therefore h will be continuous.

Now, $wh : Z \times X \to Y$ is a mapping such that

$$\begin{aligned} wh(z, x) &= w(h(z, x)) = w(g^*(z), x) \\ &= g^*(z)\,(x) = g(z, x). \end{aligned}$$

Hence wh is equivalent to the map g. Since w and h are both continuous, therefore wh is also continuous. That is, g is continuous. Hence $\mathcal{G}$ is an admissible topology.

Theorem 4.4 — If $\mathcal{G}$ and $\mathcal{U}$ are proper and admissible topologies respectively on Y^{mX}, then $\mathcal{G} \subset \mathcal{U}$.

PROOF : Since $\mathcal{U}$ is an admissible topology, the evaluation mapping $w : (Y^{mX}, \mathcal{U}) \times X \to Y$ is continuous. Since $\mathcal{G}$ is proper, the continuity of w implies the continuity of $w^* : (Y^{mX}, \mathcal{U}) \to (Y^{mX}, \mathcal{G})$. It can easily be seen that w^* is an identity mapping. Hence $\mathcal{G} \subset \mathcal{U}$.

Theorem 4.5 — Let X be regular and Y be arbitrary then for any open cover σ of X, the σ-topology on the family $C_m(X, Y)$ of all continuous multifunctions from X to Y is always admissible.

PROOF : We shall prove that the evaluation mapping $w : C_m(X, Y) \times X \to Y$ is continuous. Let $f \in C_m(X, Y)$, $x \in X$ and let V be an open subset of Y such that $w(f, x) \subset V$. That is, $f(x) \subset V$. Then $x \in f^+(V)$. Since X is regular and f is continuous, there exists an open neighbourhood U of x such that $x \in U \subset \bar{U} \subset f^+(V)$ and also such that $\bar{U}$ is contained in some member of σ. Since $f(x) \subset V$ for all $x \in \bar{U}$, therefore, $f \in (\bar{U}, V)_2$. The set $(\bar{U} \times V)_2 \times U$ is then open neighbourhood of (f, x) such that $(g, z) \in w^+(V)$ for all $(g, z) \in (\bar{U}, V)_2 \times U$. Thus w is upper semi-continuous. It can similarly be proved that w is lower semi-continuous. Hence w is continuous and the σ-topology is admissible.

Theorem 4.6 — The compact open topology $\mathcal{C}$ for Y^{mX} is proper.

PROOF : Let Z be a topological space and let $g : Z \times X \to Y$ be any continuous multifunction. We shall prove that $g^* : Z \to (Y^{mX}, \mathcal{C})$ is also continuous. Let $z \in Z$ and let $g^*(z) \in W_1(K_1, U) \cap W_2(K_2, V)$, where K_1 and K_2 are compact subsets of X and U, V are open subsets of Y. Then $g^*(z)(K_2) \subset V$ and $g^*(z)(x) \cap U \neq \phi$ for all $x \in K_1$. But $g^*(z)(K_2) \subset V \Rightarrow \{z\} \times K_2 \subset g^+(V)$. For each $x \in K_2$ let U_x be an open subset of X such that $(z, x) \in U_x \subset g^+(V)$. The family $\{U_x : x \in X\}$, being an open covering of K_2, admits of a finite subcovering, say U_{x_1}, U_{x_2}, ..., U_{x_n}. Let $\Pi_z : Z \times X \to Z$ denote the projection map on Z, then $\bigcap_{i=1}^{n} \Pi_z(U_{x_i})$ is an open neighbourhood of z such that for each $z_0 \in \bigcap_{i=1}^{n} \Pi_z(U_{x_i})$, $z_0 \times K_2 \subset g^+(V)$ or $g^*(z_0)(K_2) \subset V$. Also, $g^*(z)(x) \cap U \neq \phi$ for all $x \in K_1$ implies that $\{z\} \times K_1 \subset g^-(U)$. Then, as before, there exists an open neighbourhood U_1 of z such that $z_0 \times K_1 \subset g^-(U)$ for all $z_0 \in U_1$. Thus $U_2 = (\bigcap_{i=1}^{n} \Pi_z(U_{x_i})) \cap U_1$ is an open neighbourhood of z such that $g^*(U_2) \subset W_1(K_1, U) \cap W_2(K_2, V)$. Hence g^* is continuous.

Theorem 4.7 — Let σ_1 and σ_2 be two open coverings of Y. If σ_1 is a refinement of σ_2 then the σ_1-topology is less than or equal to the σ_2-topology. If Y is regular, the greatest lower bound of two σ-topologies on the family $C_m(X, Y)$ of all continuous multifunctions from X to Y is admissible.

PROOF : The first part of the theorem needs no proof. To prove the second part, let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two σ-topologies for $C_m(X, Y)$. Let $\mathcal{J}_1$ be corresponding to an open cover σ_1 and $\mathcal{J}_2$ be corresponding to an open cover σ_2 of X. Let $\sigma_3 = \{U \cap V : U \in \sigma_1 \text{ and } V \in \sigma_2\}$. Obviously, σ_3 is an open cover of Y which is a refinement of both σ_1 and σ_2, let $\mathcal{J}_3$ be the σ-topology for $C_m(X, Y)$ corresponding to the open cover σ_3 of X. Then $\mathcal{J}_3$ is admissible, in view of Theorem 4.5. Also $\mathcal{J}_3$, contained in the greatest lower bound of $\mathcal{J}_1$ and $\mathcal{J}_2$ is admissible.

5. NEAR TOPOLOGY

The concept of near-topology for a family of single-valued mappings Y^X has been introduced by Irudayanathan (1967).

Let Ω be the collection of all open coverings of Y. Then the topology generated by the family of all sets of the form $W(f, \alpha) = \{g \in Y^X :$ for each $x \in X$ there exists a $U_x \in \alpha$ such that $f(x), g(x) \in U_x\}$, where $f \in Y^X$ and $\alpha \in \Omega$, is called the near topology for Y^X. As usual we have the following generalisation of near topology to include the family of all multifunctions from X to Y.

Definition 5.1 — Let Ω be the collection of all open coverings of Y. For $\alpha \in \Omega$ and $f \in Y^{mX}$, let $C_1(f, \alpha) = \{g \in Y^{mX}$: for each $x \in X$ there exists a $U_x \in \alpha$ such that $f(x) \cap U_x \neq \phi$ and $g(x) \cap U_x \neq \phi\}$ and $C_2(f, \alpha) = \{g \in Y^{mX}$: for each $x \in X$ there exists a $U_x \in \alpha$ such that $f(x) \subset U_x$ and $g(x) \subset U_x\}$. Let $\mathcal{P}_n$ be the topology on Y^{mX} generated by the sub-base $\{C_1(f, \alpha), C_2(f, \alpha) : f \in Y^{mX}, \alpha \in \Omega\}$. Then $\mathcal{P}_n$ is called the near topology for Y^{mX}.

Theorem 5.2 — Let $\mathcal{F}$ be the family of all one-one point closed multifunctions from X to a regular space Y. Then $(\mathcal{F}, \mathcal{P}_n)$ is finer than $\mathcal{P}$.

PROOF : Let $f \in \Pi_x^-(U_x)$, so that $f(x) \cap U_x \neq \phi$. Let $y \in f(x) \cap U_x$. Since Y is regular, there exists an open set U_y containing y such that $y \in U_y \subset \bar{U}_y \subset U_x$. Let $h : X \to Y$ be defined as

$$h(t) = f(t) \text{ for } t \neq x,$$

$$h(x) = \bar{U}_y \cap f(x).$$

Obviously, h is a one-one point-closed multifunction. Let $z \notin \bar{U}_y$. Then there is an open subset W_z containing z such that $W_z \cap U_y = \phi$. As W_z is open, we have $W_z \cap \bar{U}_y = \phi$, that is, $W_z \cap (\bar{U}_y \cap f(x)) = \phi$. Let $\alpha = U_x \cup \{W_z : z \notin \bar{U}_y\}$. Then α is an open cover of Y such that $f \in C_1(h, \alpha) \subset \Pi_x^-(U_x)$. The fact that $f \in C_1(h, \alpha)$ is clear from the definition of h. Let now $g \in C_1(h, \alpha)$ then for $x \in X$, there exists a member of α which intersects both $g(x)$ and $h(x)$. But no member of α except U_x intersects $h(x)$, therefore $g(x) \cap U_x \neq \phi$. Hence $g \in \Pi_x^-(U_x)$.

On the other hand, if $\Pi_x^+(U_x)$ be any other $\mathcal{P}$-open set then $f \in C_2(f, \beta) \subset \Pi_x^+(U_x)$, where $\beta = \{Y \sim f(x), U_x\}$. Hence $\mathcal{P} \subset \mathcal{P}_n$.

For the converse implication, we have the following easily proved theorem.

Theorem 5.2 — For the family $\mathcal{F}$ of all constant multifunctions $\mathcal{P}_n \subset \mathcal{P}$.

Theorem 5.3 — The near topology $\mathcal{P}_n$ is contained in the topology Γ_{m_2} for the set of constant multifunctions.

PROOF : Let $C_1(f, \alpha)$ be a $\mathcal{P}_n$- subbasic open set and let $h \in C_1(f, \alpha)$. Then for each $x \in X$ there exists a $U_x \in \alpha$ such that $h(x) \cap U_x \neq \phi$ and $f(x) \cap U_x \neq \phi$. Let $V_x = X \times U_x$ then V_x is an open subset of $X \times Y$. Let $F_{V_x}^- = \{f : (\{x\} \times f(x)) \cap V \neq \phi$ for all $x \in X\}$. Then $F_{V_x}^-$ is a Γ_{m_2}-open set such that $h \in F_{V_x}^- \subset C_1(f, \alpha)$. For, let $y \in h(x) \cap U_x$, then $(x, y) \in X \times U_x = V_x$. Thus $(\{x\} \times h(x)) \cap V_x \neq \phi$. Hence $h \in F_{V_x}^-$. Now, let $g \in F_{V_x}^-$, so that for each $x \in X$, $(\{x\} \times g(x)) \cap V_x \neq \phi$.

Let x' be any point of X, and let $y' \in g(x')$ such that $(x', y') \in V_x$, that is $y' \in U_x$. Therefore $g(x') \cap U_x \neq \phi$. Since f is a constant function, $f(x) \cap U_x \neq \phi \Rightarrow f(x') \cap U_x \neq \phi$. Thus $g \in C_1(f, \alpha)$. Hence $h \in F^-_{V_x} \subset C_1(f, \alpha)$.

Now, let $C_2(g, \beta)$ be another $\mathcal{P}_n$-subbasic open set containing $h \in \mathcal{F}$. For each $x \in X$, there exists a $U_x \in \beta$ such that $h(x), g(x) \subset U_x$. Let $U = X \times U_x$. Then U is an open subset of $X \times Y$. Let $F^+_u \subset \{f \in \mathcal{F} : G(f) \subset U\}$. Then $h \in F^+_u \subset C_2(g, \beta)$, which implies that $C_2(g, \beta)$ is a Γ_{m2}-open set. Hence the near topology is contained in the Γ_{m2}-topology.

Theorem 5.4 — If Y is a T_1-space, the near topology for the family $\mathcal{F}$ of all one-one functions from X to Y contains the topology $\mathcal{P}$ of pointwise convergence.

PROOF: Let $\Pi^-_x(U_x)$ be any $\mathcal{P}$-sub-basic open set containing $f \in \mathcal{F}$. Let $z \in f(x) \cap U_x$. Define $h \in \mathcal{F}$ as follows

$$h(t) = f(t),\ t \neq x$$

$$h(x) = z.$$

Let $\alpha = \{U_x, Y \sim \{z\}\}$. Then α is an open cover for Y such that $f \in C_1(h, \alpha) \subset \Pi^-_x(U_x)$, which proves that $\Pi^-_x(U_x)$ is $\mathcal{P}_n$-open. Similarly, if $\Pi^+_x(U_x)$ is a $\mathcal{P}$-sub-basic open set containing f, then $f \in C_2(f, \beta) \subset \Pi^+_x(U_x)$, where $\beta = \{U_x, Y \sim \{y\}\}$ and $y \in f(x)$. Thus $\Pi^+_x(U_x)$ is also $\mathcal{P}_n$-open. Hence $\mathcal{P} \subset \mathcal{P}_n$.

Theorem 5.5 — If Y is a Hausdorff space, then the near-topology for the family $\mathcal{F}$ of one-one point-compact multifunctions from X to Y is T_1. However, if the near topology for $\mathcal{F}$ is a T_1-space then Y is T_1.

PROOF : Let Y be a Hausdorff space and let f, g be two distinct points of $\mathcal{F}$. There exists $x \in X$ such that $f(x) \neq g(x)$. If $f(x) \not\subset g(x)$ then there exists $y \in f(x) \sim g(x)$. Since Y is Hausdorff, there are disjoint open sets U_y and $U_{g(x)}$ containing y and $g(x)$ respectively. Let $B_1 = \{f \in \mathcal{F} : f(x) \cap U_y \neq \phi\}$ and $B_2 = \{f \in \mathcal{F} : f(x) \subset U_{g(x)}\}$. Then B_1 and B_2 are $\mathcal{P}$-open sets such that $f \in B_1$, $g \in B_2$ and $f \notin B_2$, $g \notin B_1$. By Theorem 5.4, it follows that B_1 and B_2 are $\mathcal{P}_n$-open sets. In case $g(x) \not\subset f(x)$ it can similarly be proved that there exists a neighbourhood of f not containing g and a neighbourhood of g not containing f. Hence $(\mathcal{F}, \mathcal{P}_n)$ is a T_1-space.

To prove the second part, let $a, b \in Y$ be two distinct points of Y. Define two functions f, g as follows :

$$f(x) = a,\ f(t) = g(t), \text{ for all } t \neq x$$

$$g(x) = b.$$

Then $f \neq g$ and $f, g \in \mathcal{F}$. Since $\mathcal{F}$ is a T_1-space, there exists an open set $(\bigcap_{i=1}^{n_1} C_1(f_i, \alpha_i)) \cap (\bigcap_{j=1}^{m_1} C_2(f_j, \alpha_j))$ containing f but not g and an open set $(\bigcap_{i=1}^{n_2} C_1(g_i, \beta_i)) \cap (\bigcap_{j=1}^{m_2} C_2(g_j, \beta_j))$ containing g but not f. For each $i = 1, 2, ..., n_1$, let U_{x_i} be a member of α_i such that $f(x) \cap U_{x_i} \neq \phi$ and $f_i(x) \cap U_{x_i} \neq \phi$. For each $j = 1, 2, ..., m_1$, let V_{x_j} be a member of α_j such that $f(x) \subset V_{x_j}$ and $f_j(x) \subset V_{x_j}$. Then $a \in (\bigcap_{i=1}^{n_1} U_{x_i}) \cap (\bigcap_{j=1}^{m_1} V_{x_j})$. Since $g \notin (\bigcap_{i=1}^{n_1} C_1(f_i, \alpha_i)) \cap (\bigcap_{j=1}^{m_1} C_2(f_j, \alpha_j))$, therefore $b \notin (\bigcap_{i=1}^{n_1} U_{x_i}) \cap (\bigcap_{j=1}^{m_1} V_{x_j})$. Hence there exists an open set containing a but not b. Similarly, the conditions that $g \in (\bigcap_{i=1}^{n_2} C_1(g_i, \beta_i)) \cap (\bigcap_{j=1}^{m_2} C_2(g_j, \beta_j))$ and $f \notin (\bigcap_{i=1}^{n_2} C_1(g_i, \beta_i)) \cap (\bigcap_{j=1}^{m_2} C_2(g_j, \beta_j))$ imply that there is an open set containing b but not a. Hence Y is a T_1-space.

Theorem 5.6 — If Y is a regular space then the set $\mathcal{F}$ of all one-one point-closed multifunctions from X to Y with respect to the near topology is a Hausdorff space.

PROOF : If f, g be two distinct points of $\mathcal{F}$, then there exists $x \in X$ such that $f(x) \neq g(x)$. Let $f(x) \not\subset g(x)$. Let $y \in f(x) \sim g(x)$. Since Y is regular, there are disjoint open sets U_y and V_y containing y and $g(x)$ respectively. If $B_1 = \{h \in \mathcal{F} : h(x) \cap U_y \neq \phi\}$ and $B_2 = \{h \in \mathcal{F} : h(x) \subset V_y\}$, then by Theorem 5.1. B_1 and B_2 are disjoint $\mathcal{P}_n$-open neighbourhoods of f and g respectively. Hence f, g are strongly separated and $(\mathcal{F}, \mathcal{P}_n)$ is a Hausdorff space.

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