STRESSES IN AN ELASTIC INFINITE PLATE WITH A PARABOLIC CRACK

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(Communicated by J. N. Kapur, F.N.A.)

(*Received* 8 *March* 1973)

A different and rather simple approach has been made to solve the problem of an infinite elastic plate with a parabolic crack. By making a suitable integral representation of the complex potential, the problem is reduced to Abel's type integral equation and is solved in a closed form. Stress components have been found out.

INTRODUCTION

Complex variable methods in two-dimensional classical elasticity are known (Muskhelishvilli 1963, Green and Zerna 1968). These methods have been successfully employed to solve punch and crack problems in elasticity (Paria 1957; Verma 1964, 1966; Green and Zerna 1968; Ahmed 1968, 1970, 1971). Recently, England and Green (1963) have given a simple and rather different method to solve these problems. It reduces some of the problems to the solution of simple integral equations of Abel's type which can be solved conveniently. The method is essentially similar to that used by Green and Zerna (1968) for axially symmetric problems in three-dimensional elasticity.

In this paper, the above stated method has been extended to solve the problem of an infinite elastic plate with a parabolic crack. Stress potentials have been completely determined. Stress components have been found out. The method can be employed to solve various other types of punch and crack problems.

1. STATEMENT OF THE PROBLEM

We consider the small deformation of a homogeneous, isotropic, elastic infinite plate with a parabolic crack. The region S occupied by the body is the entire z-plane cut along a parabolic are symmetrical about the vertex, whose focus is at (0, 0) and the vertex at $(0, -a^2)$. The transformation

$$z = m(\zeta) = i(\zeta - ia)^2;\ a > 0 \qquad \ldots(1.1)$$

$$z = x + iy,\ \zeta = \xi + i\eta$$

transforms the region S into S', the parabolic crack L in the z-plane into the straight crack L' in the ζ-plane along the real axis $\eta = 0$ extending from $(-\beta, 0)$ to $(\beta, 0)$. The equation to the parabola is

$$x^2 = 4a^2 (y+a^2).$$

The curve corresponding $\xi=\text{constant}=\alpha$ is a confocal parabola oriented in the opposite direction. The upper and lower segments of the crack are assumed under the action of the pressure given by

$$\left.\begin{aligned} \sigma_\eta &= -f(\xi)-g(\xi) \qquad (-\beta < \xi < \beta, \eta = 0) \\ \tau_{\xi\eta} &= 0 \qquad\qquad\qquad\quad (-\infty < \xi < \infty, \eta = 0) \end{aligned}\right\} \qquad \ldots(1.2)$$

where $f(\xi)$ and $g(\xi)$ are even and odd functions of ξ respectively. We assume that $f(\xi)$ and $g(\xi)$ are sectionally continuous in $0 \leqslant \xi \leqslant \beta$.

2. Fundamental Formulae

In order to solve the problem, we shall make use of the following fundamental formulae (Milne Thomas 1960)

$$\sigma_\xi + \sigma_\eta = 2[W(\zeta) + \overline{W}(\bar{\zeta})] \qquad \ldots(2.1)$$

$$\sigma_\eta - \sigma_\xi + 2i\,\tau_{\xi\eta} = \frac{2}{\overline{m}'(\bar{\zeta})}[\overline{m}(\bar{\zeta})\, W'(\zeta) + m'(\zeta)\, w(\zeta)] \qquad \ldots(2.2)$$

$$2\mu(u+iv) = k\phi(\zeta) - m(\zeta)\overline{W}(\bar{\xi}) - \bar{\psi}(\bar{\zeta}) \qquad \ldots(2.3)$$

where $k=(3-\nu)/(1+\nu)$, μ and ν are the rigidity modulus and Poisson's ratio of the material and functions $\phi(\zeta)$ and $\psi(\zeta)$ are stress potentials such that

$$\phi'(\zeta) = m'(\zeta)\, W(\zeta);\ \psi'(\zeta) = m'(\zeta)\, w(\zeta).$$

From eqns. (2.1) and (2.2), we may write

$$\sigma_\eta + i\tau_{\xi\eta} = W(\zeta) + \overline{W}(\bar{\zeta}) + \frac{1}{\overline{m}'(\bar{\zeta})}[\overline{m}(\bar{\zeta}) W'(\zeta) + m'(\zeta) w(\zeta)] \qquad \ldots(2.4)$$

$$\sigma_\xi - i\tau_{\xi\eta} = W(\zeta) + \overline{W}(\bar{\zeta}) - \frac{1}{\overline{m}'(\bar{\zeta})}[\overline{m}(\bar{\zeta}) W'(\zeta) + m'(\zeta) w(\zeta)]. \qquad \ldots(2.5)$$

It is convenient to introduce two other functions $L(\zeta)$ and $M(\zeta)$ by the equations

$$\left.\begin{aligned} m'(\zeta)w(\zeta) &= \overline{m}(\zeta) W'(\zeta) - L(\zeta) \\ m'(\zeta)w(\zeta) &= \overline{m}(\zeta) W'(\zeta) + M(\zeta) \end{aligned}\right\} \qquad \ldots(2.6)$$

so that

$$\sigma_\eta + i\tau_{\xi\eta} = W(\zeta) + \overline{W}(\bar{\zeta}) + \frac{1}{\overline{m'(\zeta)}}\left[\left(\overline{m(\zeta)} - \overline{m}(\zeta)\right) W'(\zeta) + M(\zeta)\right] \qquad ...(2.7)$$

$$\sigma_\xi - i\tau_{\xi\eta} = W(\zeta) + \overline{W}(\bar{\zeta}) - \frac{1}{\overline{m'(\zeta)}}\left[\left(\overline{m}(\bar{\zeta}) + \overline{m}(\zeta)\right) W'(\zeta) - L(\zeta)\right]. \qquad ...(2.8)$$

3. Solution of the Problem

If we use the formula (2.7), the boundary conditions (1.2) can be satisfied by taking $M(\zeta)=0$ and then on the boundary $\eta = 0$ ($\zeta = \bar{\zeta}$) we have

$$\sigma_\eta = W(\zeta) + \overline{W}(\bar{\zeta}) \qquad ...(3.1)$$

provided $\eta W(\zeta)$ and $\eta W'(\zeta)$ tend to zero as $\eta \to \pm 0$. Since the crack is opened by equal and opposite pressure on each side, the resultant pressure on the entire length of the crack is zero. Hence, if in addition, the stress and rotation vanish at infinity.

$$W(\zeta) = O\,(1/\zeta^2) \qquad ...(3.2)$$

for large value of of $|\zeta|$.

In order to solve the problem specified by (3.1) and (3.2) we consider the function

$$K(\zeta) = \int W(\zeta)d\zeta = \int_0^\infty \frac{F(t) + \zeta G(t)}{\sqrt{\zeta^2 - t^2}}\, dt \qquad ...(3.3)$$

where $F(t)$ and $G(t)$ are real continuous functions of t in the interval $0 \leqslant t \leqslant \beta$ and

$$\sqrt{[\zeta^2 - t^2]} = \rho e^{i\phi} \; (-\pi \leqslant \phi\pi) \qquad ...(3.4)$$

where $\rho^2 \cos 2\phi = \xi^2 - \eta^2 - t^2$; $\rho^2 \sin 2\phi = 2\xi\eta$.

An examination of the function (3.4) shows that on the real axis $\eta = 0$

$$\left.\begin{aligned}
\sqrt{[\zeta^2 - t^2]} &= \sqrt{[\xi^2 - t^2]} && (\xi > t,\ \eta \to \pm 0) \\
\sqrt{[\zeta^2 - t^2]} &= -\sqrt{[\xi^2 - t^2]} && (\xi < -t,\ \eta \to \pm 0) \\
\sqrt{[\zeta^2 - t^2]} &= i\sqrt{[t^2 - \xi^2]} && (|\xi| < t,\ \eta \to +0) \\
\sqrt{[\zeta^2 - t^2]} &= -i\sqrt{[t^2 - \xi^2]} && (|\xi| < t,\ \eta \to -0).
\end{aligned}\right\} \qquad ...(3.5)$$

The function defined by (3.3) is regular at all points of the ζ-plane where $(\zeta^2 - t^2)$ does not vanish, that is everywhere except points $\eta = 0$, $\xi < \beta$. Since $|\zeta^2 - t^2| \geqslant |\zeta|^2 - t^2 \geqslant |\zeta|^2 - \beta^2$ for large value of $|\zeta|$, we have

$$|K(\zeta)| < \frac{1}{\sqrt{|\zeta|^2 - \beta^2}} \int_0^\beta F(t)\, dt + \frac{|\zeta|}{\sqrt{|\zeta|^2 - \beta^2}} \int_0^\beta |G(t)|\, dt$$

and (3.2) is satisfied.

Suppose now that t in (3.3) is a complex variable and that $F(t)$, $G(t)$ are regular functions of t in some simply connected region containing the line segment $0 \leqslant t \leqslant \beta$ on the real t-axis. The zeros of $\sqrt{\zeta^2 - t^2}$ are at $\pm(\xi + i\eta)$ i.e. in the first and third quadrants of the t-plane when $\xi > 0$, $\eta > 0$ or $\xi < 0$, $\eta < 0$, and in the second and fourth quadrants of the t-plane where $\xi > 0$, $\eta < 0$ or $\xi < 0$, $\eta > 0$. In these cases we can displace the path of integration in (3.3) away from the real axis into the fourth and first quadrants respectively. The function $K(\zeta)$ is then regular function of ζ in the neighbourhood of the crack $|\xi| < \beta$ on the real axis $\eta = 0$ except possibly at $\zeta = \pm\beta$ and $\zeta = 0$. Thus $K(\zeta)$ and its derivatives tend to finite limits as the point ζ approaches a point on the crack in any manner through values of η greater than or less than zero, except possibly at $\zeta = \pm\beta$ or $\zeta = 0$. By excluding the points $t = |\xi|$ ($\xi \neq \pm\beta$, $\xi \neq 0$) on the real t-axis with small semicircles whose radii tend to zero we can replace the path of integration by one along the real t-axis as in (3.3). The function $K(\zeta)$ given by (3.3) and its derivatives are then continuous on to each side $\eta > 0$, $\eta < 0$ of the crack, except at $\zeta = \pm\beta$, $\zeta = 0$. The function $K(\zeta)$ given in (3.3) is continuous at $\zeta = +\beta$, but not its derivatives.

If

$$F(t) = O(t), \quad G(t) = O(1) \qquad \ldots(3.6)$$

as $t \to 0$, then from (3.3) we can show that $K(\zeta)$, $K'(\zeta)$, $\zeta K''(\zeta)$, $\zeta K''(\zeta)$ have finite values as $\zeta \to 0$ in any manner. If we relax the condition on $G(t)$ to

$$G(t) = O(1) \qquad \ldots(3.7)$$

as $t \to 0$ then $K(\zeta)$, $\zeta K'(\zeta)$, $\zeta K''(\zeta)$ have unique finite values as $\bar{\zeta} \to 0$ in any manner but $K'(\zeta)$, $\bar{\zeta} K''(\zeta)$ do not tend to finite limits. However, this only arises when $g(\xi)$ has a finite discontinuity at $\xi = 0$ and some difficulty must be expected at the point $\zeta = 0$ of the crack in this case.

Using (2.7) and (3.5), the first condition in (3.1) yields the integral equation

$$4 \frac{d}{d\xi} \int_0^\xi \frac{F(t)}{\sqrt{\xi^2 - t^2}} \, dt = -f(\xi) \qquad (0 \leqslant \xi \leqslant \beta) \qquad \ldots(3.8)$$

$$4 \frac{d}{d\xi} \int_0^\xi \frac{\xi G(t) dt}{\sqrt{\xi^2 t^2}} = -g(\xi) \qquad (0 \leqslant \xi \leqslant \beta). \qquad \ldots(3.9)$$

The solution of these equations is straightforward (Green and Zerna 1968).

Thus

$$F(t) = -\frac{1}{2\pi}\,\frac{d}{dt}\int_0^t \frac{\xi\,d\xi}{\sqrt{t^2-\xi^2}}\int_0^\xi f(u)\,du$$

$$G(t) = -\frac{1}{2\pi}\,\frac{d}{dt}\int_0^t \frac{d\xi}{\sqrt{t^2-\xi^2}}\int_0^\xi g(u)\,du$$

and changing the order of integration, we have

$$F(t) = -\frac{1}{2\pi}\,\frac{d}{dt}\int_0^t f(u)\sqrt{t^2-u^2}\,du$$

$$= -\frac{t}{2\pi}\int_0^t \frac{f(u)}{\sqrt{t^2-u^2}}\,du \qquad \text{...(3.10)}$$

and

$$G(t) = -\frac{1}{2\pi}\,\frac{d}{dt}\int_0^t g(u)\left[\frac{\pi}{2} - \sin^{-1}\frac{u}{t}\right] du$$

$$= -\frac{1}{2\pi}\int_0^t \frac{u g(u)}{\sqrt{t^2-u^2}}\,du. \qquad \text{...(3.11)}$$

If $f(u)$ and $g(u)$ are sectionally continuous then $F(t)$ and $G(t)$ are continuous in the interval $0 \leqslant u \leqslant \beta$; the results (3.10) and (3.11) can be replaced by

$$F(t) = -\frac{t}{2\pi}\int_0^{\pi/2} f(t\sin\theta)\,d\theta \qquad \text{...(3.12)}$$

$$G(t) = -\frac{1}{2\pi}\int_0^{\pi/2} g(t\sin\theta)\sin\theta\,d\theta$$

(a) If the crack is opened by a uniform pressure P, we have

$$f(\xi) = P, \quad g(\xi) = 0$$

so that

$$F(t)=-\frac{Pt}{4},\ G(t)=0 \qquad \ldots(3.13)$$

and $$K(\zeta)=-\frac{P}{4}\left[\zeta-\sqrt{\zeta^2-\beta^2}\right]. \qquad \ldots(3.14)$$

(b) Again if the crack is opened by uniform pressure P over one half $0\leqslant\xi\leqslant\beta$ and zero pressure over the rest of the crack $(-\beta\leqslant\xi<0)$ we have

$$F(t)=-\frac{P}{8}t, \qquad G(t)=-\frac{P}{4\pi} \qquad \ldots(3.15)$$

and

$$K(\zeta)=-\frac{P}{8}\left[\zeta-\sqrt{\zeta^2-\beta^2}\right]-\frac{P\zeta}{4\pi}\sin^{-1}\frac{\beta}{\zeta}. \qquad \ldots(3.16)$$

Thus the stress potentials have been determined explicitly in the two cases.

4. Expressions for Stress Components

The expression for stress and displacement components can be very easily found out with the help of the formulae (2.1)–(2.3). The stress components in the case (a) studied above when the crack is opened by a uniform pressure of magnitude P are as follows :

$$(\sigma_\xi+\sigma_\eta)/P = [R(\xi\cos\phi-\eta\sin\phi)-1] \qquad \ldots(4.1)$$

$$(\sigma_\eta-\sigma_\xi)/P = \tfrac{1}{2}\beta GR^3[(2a^2\eta-2a\eta^2-\xi^2\eta)\sin 3\phi-(a\xi\eta+\xi\eta^2)\cos 3\phi] \qquad \ldots(4.2)$$

$$\tau_{\xi\eta}/P = \beta GR^3[(a\xi\eta+\xi\eta^2)\sin 3\phi+(2a^2\eta-2a\eta^2-\xi^2\eta)\cos 3\phi] \qquad \ldots(4.3)$$

where

$$R=[(\xi^2-\eta^2-\beta^2)^2+4\xi^2\eta^2]^{-1/4};\ G=[\xi^2+(a-\eta)^2]^{-1}$$

and

$$\phi=\tfrac{1}{2}\tan^{-1}\left(\frac{2\xi\eta}{\xi^2-\eta^2-\beta^2}\right).$$

The stress components on the line $\eta=0$ $(\xi>\beta)$ are

$$\sigma_\xi/P=\frac{\sigma_\eta}{P}=\tfrac{1}{2}\left[\frac{\xi}{\sqrt{\xi^2\beta^2}}-1\right];\ \tau_{\xi\eta}=0.$$

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