UNSTEADY MOTION OF A SECOND ORDER FLUID BETWEEN PARALLEL PLATES

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The present paper deals with the unsteady motion of a non-Newtonian fluid between two infinitely extended parallel plates when upper plate is moving with uniform velocity and the lower plate is performing linear oscillations in its own plane. The technique of Laplace transform has been employed to obtain the velocity distribution, which has been shown graphically in different cases. Two particular cases have also been derived corresponding to (i) when the lower plate is at rest and (ii) when the lower plate is oscillating in absence of the upper plate.

INTRODUCTION

The exact solutions of Navier-Stokes' equations for the flow near a plate with impulsive and simple harmonic motion have been discussed by Pai (1956). The unsteady motion of viscous incompressible fluid due to periodic pressure gradient in different geometries has been investigated by Sexel (1930), Uchida (1956), Verma (1960), Drake (1965), Dube (1969) and Gupta and Goyal (1971). The problem of unsteady flow and temperature distribution of a viscous incompressible fluid between parallel plates has recently been considered by Verma and Gaur (1972) in which they have improved the results of Gupta and Goyal.

Unsteady flow of non-Newtonian fluids, which is of vital interest, has also been investigated by several researchers. Following Rosenblat (1960), who investigated the flow induced in a viscous fluid from small torsional oscillations of two infinite discs when (i) one disc is oscillating and the other is at rest and (ii) both discs oscillate with same frequency but with phase difference of 180°. Rajeswari (1961) extended the analysis for Reiner-Rivlin fluid. Bhatnagar and Rajeswari (1962) and Srivastava (1963) have studied the same problem for a special class of the Rivlin-Ericksen 'second order' fluid while Frater (1964) has discussed only the first case for Oldroyd fluid. Bhatnagar and Rajeswari have found that a reversal of the direction of the steady secondary flow is a characteristic feature of the Rivlin-Ericksen fluid. This phenomenon has also been predicted by Frater for Oldroyd fluid with elastic parameter $\sigma < (1/3)$ and a critical range of fluid parameter S. Verma and Rajvanshi (1968) have

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studied the same problem (both the cases being discussed) for Maxwell fluid and they have observed no reversal of flow contrary to the reversal of flow as established by Frater for Oldroyd fluid.

In the present paper we have considered the unsteady flow of a second order fluid between two parallel infinitely extended plates when upper plate is in uniform motion, with velocity U and the lower one is performing linear oscillations, in its own plane, of the type $u = Ve^{int}$. The method based on Laplace transform is used to obtain the velocity distribution. The velocity profiles have been plotted in different cases to show the effect of oscillations; so also the deviation of non-Newtonian fluid from Newtonian fluid. Coefficient of skin friction at the lower plate is also plotted.

Equations of Motion

The constitutive equation of an incompressible second order fluid has been given by Coleman and Noll (1960) as

$$\tau_{ij} = \phi_1 \; A_{(1)ij} + \phi_2 \, A_{(2)ij} + \phi_3 \, A_{(1)ik} \, A_{(2)kj} \qquad \ldots(1)$$

where

$$A_{(1)ij} = v_{i,j} + v_{j,i} \qquad \ldots(2)$$

$$A_{(2)ij} = a_{i,j} + a_{j,i} + 2\, v_{m,i}\, v_{m,j} \qquad \ldots(3)$$

and

$$S_{ij} = \tau_{ij} - pg_{ij} \qquad \ldots(4)$$

such that S_{ij} is the stress tensor; g_{ij} the metric tensor; v_i and a_i the velocity and acceleration vectors respectively; p the pressure; ϕ_1, ϕ_2 and ϕ_3 material constants—the coefficient of ordinary viscosity, the coefficient of visco-elasticity and the coefficient of cross viscosity respectively.

The momentum equations for the unsteady incompressible flow are

$$\rho\left(\frac{\partial v_i}{\partial t} + v^j\, v_{i,j}\right) = -\, p_{,i} + \tau^j{}_{i,j} \qquad \ldots(5)$$

and the equation of continuity is

$$v^i{}_{,i} = 0 \qquad \ldots(6)$$

where v_i is the velocity vector, $\tau^j{}_i$ the stress tensor, p the pressure, ρ the density and comma denotes covariant differentiation.

Transforming all tensor components in eqns. (1) to (6) into physical components in Cartesian coordinates, we get the following equations for the unsteady two dimensional flow :

Equation of Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad \ldots(7)$$

Momentum Equations

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\phi_1\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$+\phi_2\left[\frac{\partial^3 u}{\partial t\,\partial x^2}+\frac{\partial^3 u}{\partial t\,\partial y^2}+2\frac{\partial u}{\partial x}\left(5\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+3\frac{\partial u}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\,\partial y}\right)\right.$$

$$+2\frac{\partial v}{\partial x}\left(\frac{\partial^2 u}{\partial x\,\partial y}+2\frac{\partial^2 v}{\partial x^2}\right)+\frac{\partial v}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}+3\frac{\partial^2 v}{\partial x\,\partial y}\right)+u\left(\frac{\partial^3 u}{\partial x^3}\right.$$

$$\left.\left.+\frac{\partial^3 u}{\partial x\,\partial y^2}\right)+v\left(\frac{\partial^3 u}{\partial x^2\,\partial y}+\frac{\partial^3 u}{\partial y^3}\right)\right]+\phi_3\left[8\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right.$$

$$\left.\times\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)\right] \qquad \ldots(8)$$

and

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\phi_1\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$+\phi_2\left[\frac{\partial^3 v}{\partial t\,\partial x^2}+\frac{\partial^3 v}{\partial t\,\partial y^2}+2\frac{\partial v}{\partial y}\left(5\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\right)+3\frac{\partial v}{\partial x}\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2}\right)\right.$$

$$+2\frac{\partial u}{\partial y}\left(\frac{\partial^2 v}{\partial y\,\partial x}+2\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial x}\left(3\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)$$

$$\left.+u\left(\frac{\partial^3 v}{\partial x\,\partial y^2}+\frac{\partial^3 v}{\partial x^3}\right)+v\left(\frac{\partial^3 v}{\partial y^3}+\frac{\partial^3 v}{\partial y\,\partial x^2}\right)\right]+\phi_3\left[8\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\right.$$

$$\left.+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)\right]. \qquad \ldots(9)$$

FORMULATION OF THE PROBLEM

We consider the flow of a second order fluid contained between two infinite parallel plates at a distance of y_0 from each other. The origin is considered to be on the lower plate along which the x-axis is taken and perpendicular to it and through origin the y-axis is considered. The motion of the second order fluid is brought about by the moments given to the two plates. Initially the upper plate is assumed to be moving with a constant velocity, U, and then from the certain instant, the lower plate starts executing simple harmonic motion, in its own plane with amplitude V and frequency n.

For the present geometry of the problem, the velocity components of the fluid and pressure are

$$u = u(y, t),\ v = 0 \text{ and } p = p(y, t). \qquad \text{...(10)}$$

From eqns. (8) to (10), we have

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial P^*}{\partial x} + \phi_1 \frac{\partial^2 u}{\partial y^2} + \phi_2 \frac{\partial^3 u}{\partial t\, \partial y^2}. \qquad \text{...(11)}$$

and

$$0 = -\frac{\partial P^*}{\partial y} \qquad \text{...(12)}$$

where P^* is the modified pressure (Bhatnagar 1961) given by

$$P^* = p - (2\phi_2 + \phi_3)\left(\frac{\partial u}{\partial y}\right)^2. \qquad \text{...(13)}$$

From (10) and (13), we have

$$\frac{\partial P^*}{\partial x} = 0.$$

and hence eqns. (11) and (12) reduce to

$$\rho \frac{\partial u}{\partial t} = \phi_1 \frac{\partial^2 u}{\partial y^2} + \phi_2 \frac{\partial^3 u}{\partial t\, \partial y^2}. \qquad \text{...(14)}$$

Method of Solution

In order to solve eqn. (14), we assume the expression for the velocity as

$$u = u_0(y) + u_1(y, t) \qquad \text{...(15)}$$

where $u_0(y)$ and $u_1(y, t)$ are respectively steady and unsteady part of the solution of eqn. (14).

Substituting (15) into (14) and equating steady and unsteady terms on both the sides, we get following differential equations:

$$\frac{\partial^2 u_0}{\partial y^2} = 0. \qquad \text{...(16)}$$

and

$$\frac{\partial u_1}{\partial t} = \nu_1 \frac{\partial^2 u_1}{\partial y^2} + \nu_2 \frac{\partial^3 u_1}{\partial t\, \partial y^2} \qquad \text{...(17)}$$

where

$$\nu_1 = \frac{\phi_1}{\rho} \; ; \; \nu_2 = \frac{\phi_2}{\rho} \, .$$

SOLUTIONS OF EQUATIONS

Equation (16) is to be solved subject to the following boundary conditions:

$$\left. \begin{aligned} u_0 &= 0 \; ; \text{at } y = 0 \\ u_0 &= U \; ; \text{at } y = y_0 . \end{aligned} \right\} \qquad \ldots(18)$$

Its solution is

$$u_0 = \frac{U}{y_0} y. \qquad \ldots(19)$$

Now in order to solve eqn. (17), we define the Laplace transform

$$\bar{u}_1 (y, p) = \int_0^\infty e^{-pt} \, u_1 (y, t) \, dt, \quad p > 0 \qquad \ldots(20)$$

Equation (17) is to be solved subject to the following boundary conditions:

$$\left. \begin{aligned} &\text{(a)} \quad u_1 = V e^{int} \; ; \quad \text{at} \quad y = 0 \; ; \quad t > 0 \\ &\text{(b)} \quad u_1 = 0 \; ; \quad \text{at} \quad y = y_0 \; ; \quad t > 0 \\ &\text{(c)} \quad u_1 = 0 \; ; \quad \text{at} \quad t = 0 \quad \text{for all } y \end{aligned} \right\} \qquad \ldots(21)$$

where real part of $V e^{int}$ is to be considered.

Multiplying both the sides of (17) by e^{-pt} and integrating between the limits 0 to ∞, we have the Laplace transform of (17) as

$$\frac{\partial^2 \bar{u}_1}{\partial y^2} - \frac{p}{\nu_1 + \nu_2 p} \bar{u}_1 = 0 \qquad \ldots(22)$$

and the transformed boundary conditions are

$$\left. \begin{aligned} &\text{(a)} \quad \bar{u}_1 = \frac{V}{p - in} \quad \text{at } y = 0 \\ &\text{(b)} \quad \bar{u}_1 = 0 \quad \text{at } y = y_0 . \end{aligned} \right\} \qquad \ldots(23)$$

Now the solution of (22) subject to (23) is

$$\bar{u}_1 = \frac{V \sinh \left[\left(\dfrac{p}{\nu_1 + \nu_2 p} \right)^{1/2} (y_0 - y) \right]}{(p - in) \sinh \left[\left(\dfrac{p}{\nu_1 + \nu_2 p} \right)^{1/2} y_0 \right]} . \qquad \ldots(24)$$

The inverse Laplace integral of (24) is evaluated by transforming the path of integration into a closed contour and applying the calculus of residues (Carslaw and Jaegar 1941). We obtain

$$u_1 = \frac{V e^{int} \sinh [\lambda_0 (y_0 - y)]}{\sinh (\lambda_0 y_0)}$$

$$+ \sum_{k=1}^{\infty} V (-1)^k 2\pi k \nu_1 \sin \left[k\pi \left(1 - \frac{y}{y_0} \right) \right]$$

$$\times \exp \left[- t \left(\frac{\nu_1 k^2 \pi^2}{y_0^2 + \nu_2 k^2 \pi^2} \right) \right] \Big/ \left(1 + \frac{\nu_2 k^2 \pi^2}{y_0^2} \right)$$

$$\times [\nu k^2 \pi^2 + in (y_0^2 + \nu_2 k^2 \pi^2)] \qquad \ldots(25)$$

where

$$\lambda_0 = \sqrt{\frac{n (\nu_2 n + i \nu_1)}{\nu_1^2 + n^2 \nu_2^2}}.$$

Collecting the real part of (25) and from (19), eqn. (15) becomes

$$u = \left(\frac{U}{y_0} \right) y + V \Big[\sinh A \cos B \Big\{ \sinh A \left(1 - \frac{y}{y_0} \right) \cos B \left(1 - \frac{y}{y_0} \right)$$

$$\times \cos nt - \cosh A \left(1 - \frac{y}{y_0} \right) \sin B \left(1 - \frac{y}{y_0} \right) \sin nt \Big\} + \cosh A$$

$$\times \sin B \Big\{ \sinh A \left(1 - \frac{y}{y_0} \right) \cos B \left(1 - \frac{y}{y_0} \right) \sin nt + \cosh A \left(1 - \frac{y}{y_0} \right)$$

$$\times \sin B \left(1 - \frac{y}{y_0} \right) \cos nt \Big\} \Big] \Big/ \Big\{ \sinh^2 A \cos^2 B + \cosh^2 A \sin^2 B \Big\}$$

$$+ V \sum_{k=1}^{\infty} (-1)^k 2 k^3 \pi^3 \sin k\pi \left(1 - \frac{y}{y_0} \right)$$

$$\times \exp \left(- \frac{nt k^2 \pi^2}{2 \alpha^2 + \beta k^2 \pi^2} \right) \Big/ \left(1 + \frac{\beta k^2 \pi^2}{2 \alpha^2} \right) \left[k^4 \pi^4 + \alpha^4 \left(1 + \frac{\beta k^2 \pi^2}{2 \alpha^2} \right)^2 \right] \qquad \ldots(26)$$

where

$$A = \alpha \left[\frac{\beta + \sqrt{(1 + \beta^2)}}{1 + \beta^2} \right]^{1/2}$$

$$B = \frac{\alpha}{[(1 + \beta^2) \{ \beta + \sqrt{(1 + \beta^2)} \}]^{1/2}}$$

$$\alpha = (\sqrt{(n/2\nu_1)})\, y_0 \text{ and } \beta = (n\, \nu_2/\nu_1).$$

On putting $\beta = 0$, the velocity expression (26) reduces to give Newtonian case obtained by Sacheti and Bhatt (1973). Two particular cases may be derived from (26).

(i) When the lower plate is kept at rest i.e. $V=0$, the well known Couette flow is obtained.

We get from eqn. (26)

$$u = \left(\frac{U}{y_0}\right) y. \qquad \text{...(27)}$$

(ii) When $y_0 \to \infty$ and $U = 0$, eqn. (26) in reduced to

$$\frac{u}{V} = e^{-A_1 y} \cos (nt - B_1 y), \qquad \text{...(28)}$$

where

$$A_1 = (\sqrt{(n/2\nu_1)}) \left[\frac{B + \sqrt{(1+\beta^2)}}{1+\beta^2} \right]^{1/2}$$

and

$$B_1 = \sqrt{(n/2\nu_1)}\ [(1+\beta^2)(\beta + \sqrt{(1+\beta^2)})]^{1/2}$$

which is the flow of second order fluid caused by an infinite oscillating plate.

Coefficient of Friction

Coefficient of friction is given by the following expression

$$c_f = \frac{y_0}{U}\left(\frac{\partial u}{\partial y} + \frac{\mu_2}{\mu_1} \frac{\partial^2 u}{\partial t\, \partial y} \right). \qquad \text{...(29)}$$

Thus at the lower plate, we have from (26) and (29)

$$(c_f)_{y=0} = 1 + \frac{\begin{array}{l}\varepsilon[\sinh A \cos B \{ P' (\cos nt - \beta \sin nt) + Q' (\sin nt \\ + \beta \cos nt)\} + \cosh A \sin B \{P' (\sin nt + \beta \cos nt) \\ + Q' (\beta \sin nt - \cos nt)\}]\end{array}}{\sinh^2 A \cos^2 B + \cosh^2 A \sin^2 B}$$

(*equation continued p.* 1003)

$$- \varepsilon \sum_{k=1}^{\infty} \left[\frac{2 k^4 \pi^4 \exp\left(- \frac{k^2 \pi^2 nt}{2 \alpha^2 + \beta k^2 \pi^2} \right) \left(1 - \frac{\beta k^2 \pi^2}{2 \alpha^2 + \beta^2 k^2 \pi^2} \right)}{\left(1 + \frac{\beta k^2 \pi^2}{2 \alpha^2} \right) \left[k^4 \pi^4 + \alpha^4 \left(1 + \frac{\beta k^2 \pi^2}{2\alpha^2} \right)^2 \right]} \right] \quad \ldots(30)$$

where

$$\varepsilon = \frac{V}{U}$$

$$P' = B \sinh A \sin B - A \cosh A \cos B$$

$$Q' = B \sinh A \sin B + A \cosh A \cos B.$$

Numerical Discussion

The velocity profiles have been shown graphically in Figs. 1 to 4 for $\varepsilon = 0.5$ and $\varepsilon = 1$. Figs. 1 and 2 represent the velocity profiles plotted against y/y_0 for different values of nt. The oscillations set up by the lower plate are affected by the motion of the upper plate. The resulting motion, as expected, is the superimposition of the motion caused by the oscillating plate and the Couette flow. Figs. 3 and 4 represent the behaviour of the velocity with respect to nt for different values of y/y_0. As we move from lower plate towards upper plate, the oscillations are damped. Effect of non-Newtonian parameter has been shown in each case. The coefficient of skin friction at the lower plate has also been plotted against nt (Fig. 5) and is maximum for $nt = 3\pi/4$, while at $nt = 0$, it gives the value for plane Couette flow.

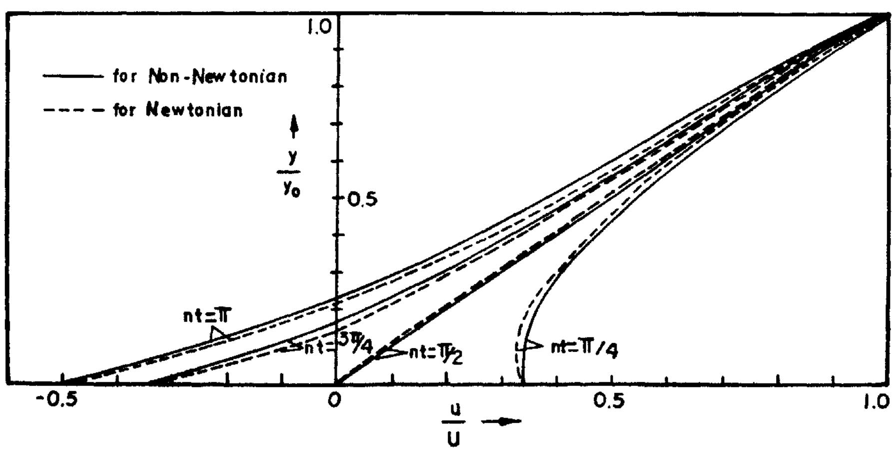

Fig. 1. Velocity profiles plotted against y/y_0 for $\varepsilon = 0.5$, $\alpha = 4$ and $\beta = -1/3$.

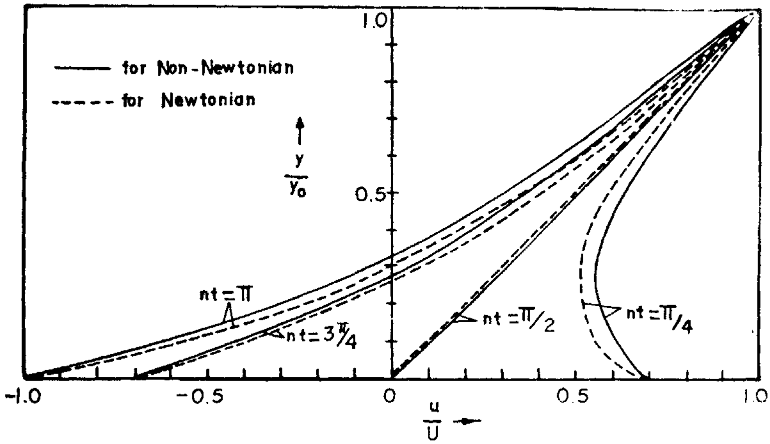

FIG. 2. Velocity profiles plotted against y/y_0 for $\varepsilon = 1$, $\alpha = 4$ and $\beta = -1/3$.

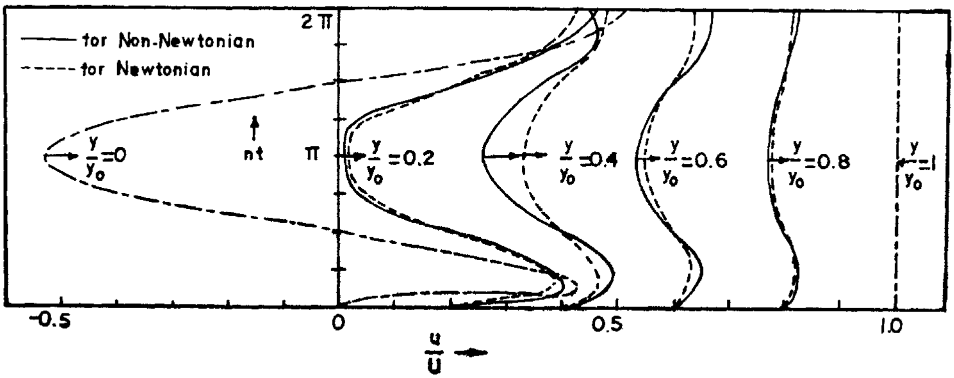

FIG. 3. Velocity profiles plotted against nt for $\varepsilon = 0.5$, $\alpha = 4$ and $\beta = -1/3$.

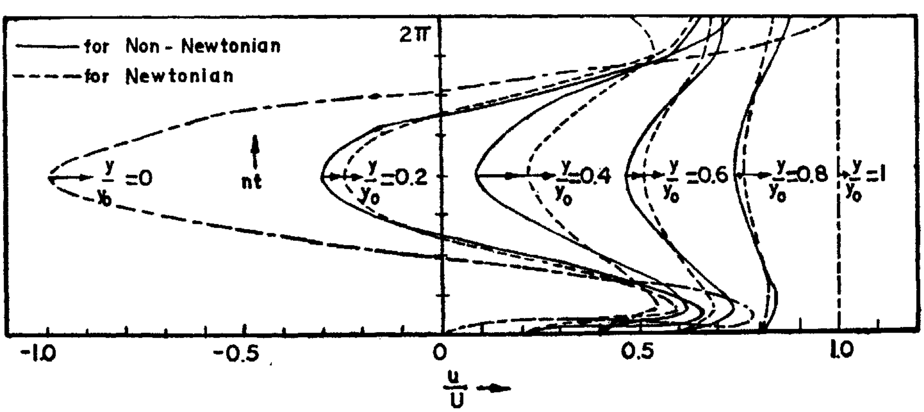

FIG. 4. Velocity profiles plotted against nt for $\varepsilon = 1$, $\alpha = 4$ and $\beta = -1/3$.

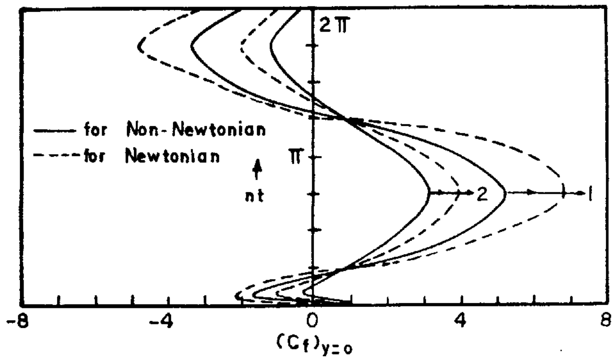

FIG. 5. Coefficients of skin friction plotted against nt where $\alpha = 4$ and $\beta = -1/3$; (1) for $\varepsilon = 1$ and (2) for $\varepsilon = 0.5$.

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