NEARLY COMPACT SETS AND THE FARTHEST POINT MAP

by T. D. NARANG, *Department of Mathematics, University of Delhi, Delhi*

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In this paper, a study of various types of continuity of the farthest point map in metric spaces has been made. These results turn out to be the exact analogues of the known results for the nearest point map in the theory of nearest points.

1. INTRODUCTION

In the theory of farthest points, the notion of nearly compact set (Ahuja *et al.* 1977) [also called ($\underline{\wedge}$) compact (Blatter 1969) or *M*-compact (Panda and Kapoor 1977)] is analogous to the notion of approximatively compact set, introduced by Efimov and Stechkin (1958), in the theory of nearest points. It was proved by Ahuja *et al.* (1975) that if K is a compact set in a metric space and is uniquely remotal, then the farthest point map is continuous. For normed linear spaces, this was proved by Motzkin *et al.* (1953). Theorem 3 of this paper proves this result for nearly compact sets. In the particular case, when the underlying space is a Banach space, this was proved by Blatter (1969). Theorem 1 studies the upper (K)-semicontinuity and Theorem 2 studies the upper semi-continuity of the farthest point map. It turns out that these results are exact analogues of the known results for the nearest point map in the theory of nearest points.

To start with, we recall the following definitions :

If G is a remotal set (Ahuja *et al.* 1977) in a metric space (X, d), then the set-valued mapping φ, which associates with each point $p \in X$, the set

$$\varphi(p) = \{y \in G : d(p, y) = \sup_{x \in G} d(p, x)\}$$

of points of G farthest from p is called the farthest point map. Thus, φ maps X into 2^G, where 2^G denotes the collection of all subsets of G. If G is uniquely remotal (Ahuja *et al.* 1977), then φ is single-valued.

If E and F are two metric spaces, then a mapping $f : E \to 2^F$ is called upper (K)-semicontinuous, if the relations

$$\lim_{n \to \infty} x_n = x, \; y_n \in f(x_n), \; \lim_{n \to \infty} y_n = y$$

imply $y \in f(x)$.

f is called upper-semicontinuous if the set $\{x \in E : f(x) \subseteq M\}$ is open for every open subset M of F.

2. Upper (K)-Semicontinuity of the Farthest Point Map

In the theory of nearest points, we know that if G is an arbitrary proximinal set in a metric space (X, d), then the nearest point mapping maps X into 2^G and is upper (K)-semicontinuous (Singer 1970). The following theorem is an analogue of this result for the farthest point map.

Theorem 1 — If G is a closed and remotal set in a metric space (X, d), then the farthest point mapping φ maps X into 2^G and is upper (K)—semicontinuous.

Proof: Let $\langle x_n \rangle$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$, $y_n \in \varphi(x_n)$ and $\lim_{n\to\infty} y_n = y$. Since G is closed, $y \in G$. We claim that $y \in \varphi(x)$. $y_n \in \varphi(x_n)$ implies $d(x_n, y_n) = \sup_{z\in G} d(x_n, z)$. Therefore, $\lim d(x_n, y_n) = \lim \sup_{z\in G} d(x_n, z)$ gives $d(x, y) = \sup_{z\in G} d(x, z)$, implying that $y \in \varphi(x)$. Hence, φ is upper (K)-semicontinuous.

3. Upper Semicontinuity of the Farthest Point Map

It is known that if E is a metric space and G, a non-void approximatively compact set in E, then the nearest point mapping maps E into 2^G and is upper semicontinuous (Singer 1970). The following theorem is an analogue to this result for the farthest point map. In the particular case, when E is a normed linear space, this result has been proved by Blatter (1969).

Theorem 2 — If G is a non-void nearly compact set in a metric space (X, d), then the farthest point mapping φ maps X into 2^G and is upper semicontinuous.

Proof: Let W be an open subset of X and let $M = \{z \in X : \varphi(z) \subseteq W\}$. We claim that M is open. Let a sequence $\langle z_n \rangle$ in X be such that $z_n \to z_0$, where $z_0 \in M$. We show that $\langle z_n \rangle$ is eventually in M. Suppose it is not true, then there exists a subsequence $\langle z_{n_i} \rangle$ of $\langle z_n \rangle$ such that $\varphi(z_{n_i}) \cap W' \neq \phi$, W' denotes the complement of W.

Let $y_{n_i} \in \varphi(z_{n_i}) \cap W'$, then $d(z_{n_i}, y_{n_i}) = \sup_{z\in G} d(z_{n_i}, z)$

and since $| d(z_{n_i}, y_{n_i}) - d(z_0, y_{n_i}) | \leqslant d(z_{n_i}, z_0)$, we see that

$$d(z_0, y_{n_i}) \to \sup_{z\in G} d(z_0, z).$$

Thus, $\langle y_{n_i} \rangle$ is a maximizing sequence in G for the element z_0 and by the definition of G, it has a convergent subsequence. Suppose y_0 is a cluster point of the sequence $\langle y_{n_i} \rangle$. Then $y_0 \in \varphi(z_0)$ and since W' is closed, $y_0 \in W'$. Thus,

$y_0 \in \varphi(z_0) \cap W'$ and, therefore, $z_0 \notin M$, a contradiction. Hence, $\langle z_n \rangle$ is eventually in M implying that M is open.

4. Continuity of the Farthest Point Map

Blatter (1969) has shown that in a Banach space, a nearly compact and uniquely remotal set supports a continuous farthest point map. The following theorem proves this result of Blatter for an arbitrary metric space. This is an exact analogue of the result 'a Chebyshev set in a metric space, if approximatively compact, supports a continuous nearest point map (Singer 1970)', in the theory of nearest points.

Theorem 3 — If K is a nearly compact and uniquely remotal subset of a metric space (X, d), then the farthest point map is continuous.

Proof : The proof of this theorem follows from Theorem 2 using the facts that for uniquely remotal sets, the farthest point map is single-valued and for single-valued maps, the two concepts of upper-semi-continuity and continuity coincide.

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