INDUCED POTENTIAL PROBLEM IN THREE DIMENSIONS—II (PROPERTIES OF THE ADJOINT KERNEL)

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Properties of the kernel of the adjoint integral equation corresponding to the Neumann-Poincaré problem are discussed. For this, the space of continuous functions is endowed with yet another type of norm, similar to the Dirichlet norm, employed in the discussion of Part I of the paper dealing with the kernel of the integral equation, associated with Robin-Poincaré problem. (Nayar 1974).

The left symmetrizers of the kernels of the two integral equations help in establishing, in norm, a Pythagoras type result between elements under the linear transformations involved and the two norms employed. These norms are linearly comparable and so are Poincaré fundamental functions and their normal derivatives connected with two boundary value problems and the associated surface.

Hilbert's quadratic form, bilinear expansions and formal solution of the adjoint integral equation as a Mittag-Leffler series in terms of fundamental functions, conclude the paper. It is observed that the spectrum of either kernel solves both the integral equations.

1. INTRODUCTION

In an earlier paper (Nayar 1974), the properties of the kernel associated with the Robin-Poincaré problem were discussed in detail and the solution of the corresponding functional equation

$$\varphi = f + \lambda K\varphi \qquad \text{...(1.1)}$$

was given in terms of the eigen set $\{\varphi_i, \lambda_i^{-1}\}$ of K in the form of Mittag-Leffler series

$$\varphi = f + \lambda \sum \frac{f_i}{\lambda_i - \lambda} \varphi_i, \quad \lambda \neq \lambda_i. \qquad \text{...(1.2)}$$

The series is uniformly and absolutely convergent everywhere. Here, f is the given boundary value relevant to the problem and φ corresponds to the density of the single layer, the potential of which gives harmonic function sought. If r_{pq} is the distance between the points p and q on a smooth surface S and n_p is the inward drawn normal at p, then $K(p, q) = \frac{\partial}{\partial n_p} G(p, q)$, $G(p, q) = \frac{1}{2\pi} \cdot \frac{1}{r_{pq}}$. The quantities

f_i in (1.2) are the generalized Fourier coefficients $[f, \varphi_i]$ with respect to the Dirichlet norm introduced into the space H of functions that are continuous in the ordinary sense.

A functional equation similar to (1.1) arises in the case of the induced potential problem considered by Howland and Vaillancourt (1966), which deals with the potential of charges induced on a smooth surface S by the external field, the medium inside and outside the surface being of different conductivities. Solution (1.2) is directly applicable to it.

Let $X(P)$ be the external field and $V(P)$, the induced potential; then μ, the density on the surface, is given by the linear integral equation of Fredholm type

$$\mu(p) = \lambda K\mu + \lambda \frac{\partial X}{\partial n} \qquad ..(1.3)$$

where $\lambda = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$. Here, σ_1, σ_2 are the conductivities of the medium inside and outside S and n is the inward drawn normal. In the physically important case $| \lambda | < 1$.

Following the notation used earlier (Nayar 1974) and identifying φ_i with μ_i and $f = \lambda \frac{\partial X}{\partial n}$, (1.2) gives

$$\mu = \lambda \frac{\partial X}{\partial n} + \lambda^2 \sum \frac{\left[\frac{\partial X}{\partial n}, \mu_i\right]}{\lambda_i - \lambda} \mu_i \qquad ...(1.4)$$

whence

$$V(P) = G\mu = \lambda G \frac{\partial X}{\partial n} + \lambda^2 \sum \frac{\left(\frac{\partial X}{\partial n}, V_i\right)}{\lambda_i - \lambda} V_i \qquad ...(1.5)$$

solves the induced potential problem in terms of $G\mu_i = V_i$, called Poincaré fundamental functions associated with the surface S. V_i are also known to be eigenfunctions of the adjoint kernel K^*, the properties of which will now be studied on identical lines.

2. Properties of the Adjoint Kernel

The adjoint kernel arises in the solution of the Neumann-Poincaré problem or more commonly the Dirichlet problem (Kellogg 1929) concerned with finding a harmonic function in R or R', the regions interior and exterior to a smooth bounded surface S and attains given boundary value g on it. The corresponding functional equation that arises is

$$\psi = g + \lambda K^* \psi \qquad ...(2.1)$$

where $K^* = \frac{\partial}{\partial n_q} G(p, q)$ and ψ is the moment of the double distribution on S, the potential of which gives the harmonic function sought.

Let γ be the moment of the double distribution on S; then the potential

$$W[\gamma] = \frac{1}{2\pi} \int \frac{\partial}{\partial n_q} G(P, q)\, \gamma(q)\, dS_q; \quad P \in R \qquad \text{...(2.2)}$$

is harmonic in R, but has discontinuities across S, given by

$$W_+ = \gamma + K^*\gamma; \quad W_- = -\gamma + K^*\gamma \qquad \text{...(2.3)}$$

which are equivalent to the functional equation

$$\gamma = g + \lambda K^*\gamma \qquad \text{...(2.4)}$$

λ being the parameter. For the interior problem $\lambda = -1$ and $W_+ = g$ and for the exterior problem $\lambda = 1$ and $W_- = -g$.

Tauber (1897, 1898) has shown that for continuous γ, if (2.2) admits a regular normal derivative on one side of S, it does so on the other side also and the limiting values are equal. Let $D\gamma$ be the normal derivative of $W[\gamma]$ from either side; then

$$D(p, q) = \frac{\partial^2}{\partial n_p\, \partial n_q} G(p, q) \qquad \text{...(2.5)}$$

represents the normal component of the force at $p(q)$ due to a unit dipole at $q(p)$.

Howland (1968), with the help of Liapounoff's extension of Green's third identity, established that

$$GK\mu = K^*G\mu; \quad DG\mu = (K^2 - I)\,\mu \qquad \text{...(2.6)}$$

and

$$DK^*\gamma = KD\gamma; \quad GD\gamma = (K^{*2} - I)\,\gamma \qquad \text{...(2.7)}$$

where μ and γ are the density and moment of the single and double distributions, respectively. We know that G, the left symmetrizer of K (Nayar 1974), having the spectral representation $\{\mu_i, \lambda_i^{-1}\}$, easily gives in view of (2. 6), the spectral representation of K^* as $\{G\mu_i, \lambda_i^{-1}\}$.

Analogously, we may employ D, the left symmetrizer of K^* to yield the spectrum of K.

Let γ_i be the eigen-function of K^* corresponding to $\lambda_i^{-1} \neq 0$; then

$$\gamma_i = \lambda_i K^*\gamma_i \text{ and } D\gamma_i = \lambda_i DK^*\gamma_i = \lambda_i KD\gamma_i.$$

Thus, K has spectral representation $\{D\gamma_i, \lambda_i^{-1}\}$. In order to reconcile the two representations, we invoke (2.7) and can easily establish that $D\gamma_i \sim \mu_i$. Thus, we have :

Theorem 1 — If μ_i and γ_i are the eigen-functions of K and K^* corresponding to λ_i^{-1} and G and D are the left symmetrizers of the two kernels, then their spectra are given by the sets $\{\mu_i, \lambda_i^{-1}\}$ and $\{G\mu_i, \lambda_i^{-1}\}$ or $\{D\gamma_i, \lambda_i^{-1}\}$ and $\{\gamma_i, \lambda_i^{-1}\}$ respectively, where $G\mu_i = \gamma_i$ and $D\gamma_i \sim \mu_i$.

In the same strain, the following result is obtained.

Theorem 2 — The kernels K and K^* are fully symmetrizable on the left by G and D, respectively, i.e., $GK\mu = 0$ if and only if $\mu \equiv 0$ and $DK^*\gamma = 0$ if and only if $\gamma \equiv 0$, where $\gamma = G\mu$.

3. Poincaré Fundamental Functions

Let μ_i and γ_i be the eigen-functions of K and K^* corresponding to the eigen-value λ_i^{-1} and $V_i = G\mu_i$ and $W_i = W[\gamma_i]$ denote potentials due to density of single distribution μ_i and moment of double distribution γ_i on S respectively. V_i and W_i are called Poincaré's fundamental functions with respect to the surface S. The above method of symmetrizers helps in establishing a result due to Blumenfeld and Mayer (1914) rather easily. The discontinuity results yield for $\mu = \mu_i$ and $\gamma = \gamma_i$ the following

$$\mu_i = \frac{\lambda_i}{1 - \lambda_i} \cdot \frac{\partial V_i}{\partial n_+} = \frac{\lambda_i}{1 + \lambda_i} \cdot \frac{\partial V_i}{\partial n_-} \qquad \text{...(3.1)}$$

and

$$\gamma_i = \frac{\lambda_i}{1 + \lambda_i} W_{i+} = \frac{\lambda_i}{1 - \lambda_i} W_{i-}. \qquad \text{...(3.2)}$$

Since $V_i = G\mu_i = \gamma_i$, we have

$$V_i = \frac{\lambda_i}{1 + \lambda_i} W_{i+} = \frac{\lambda_i}{1 - \lambda_i} W_{i-}. \qquad \text{...(3.3)}$$

Further, in view of (2.6) and (3.1), we have

$$\frac{\partial W_i}{\partial n_+} = D\gamma_i = DG\mu_i = (K^2 - I)\,\mu_i = \frac{1 - \lambda_i^2}{\lambda_i^2}\,\mu_i = \frac{1 + \lambda_i}{\lambda_i}\,\frac{\partial V_i}{\partial n_+}. \qquad \text{...(3.4)}$$

Similarly,

$$\frac{\partial W_i}{\partial n_-} = \frac{1-\lambda_i}{\lambda_i} \cdot \frac{\partial V_i}{\partial n_-}. \qquad ...(3.5)$$

For continuous γ_i, (3.4) and (3.5) are equal. The above results may be summarized as follows:

Theorem 3 — The Poincaré's fundamental functions V_i and W_i are simple multiples of each other and so are their normal derivatives.

4. Spaces with G-Norm and D-Norm

In part I of this paper (Nayar 1974), we defined a space H of continuous functions and endowed it with the Dirichlet norm using G, the left symmetrizer of K. It was established that the transformation $K : H \to H$ is symmetric and continuous with respect to this norm. Now it is natural to look for that norm with respect to which the adjoint kernel K^* is both symmetric and continuous. We likewise employ its left symmetrizer D and define two norms in H, the G-norm, being the same as the Dirichlet norm and the D-norm.

Indicating the two norms by square brackets with subscripts and the usual scalar product by round brackets, we have

$$\| \mu \|_G^2 = [\mu, \mu]_G = (\mu, G\mu) \qquad ...(4.1)$$

and

$$\| \gamma \|_D^2 = [\gamma, \gamma]_D = (\gamma, D\gamma). \qquad ...(4.2)$$

Evidently, by virtue of (2.6) and the symmetric property of K, we have

$$\begin{aligned} \| \gamma \|_D^2 &= [\gamma, \gamma]_D = [G\mu, G\mu]_D = (G\mu, DG\mu) = (G\mu, (K^2-1)\mu) \\ &= (G\mu, K^2\mu) - (G\mu, \mu) \\ &= [\mu, K^2\mu]_G - [\mu, \mu]_G \\ &= \| K\mu \|_G^2 - \| \mu \|_G^2 \cdot \end{aligned} \qquad ...(4.3)$$

The above result will be useful in comparing the two norms. We observe in passing that Pythagoras type result exists between the elements under the linear transformations G and K and the two norms introduced, i.e., from (4.3), we have

$$\| K\mu \|_G^2 = \| G\mu \|_D^2 + \| \mu \|_G^2 \cdot \qquad ...(4.4)$$

Let γ_1 and $\gamma_2 \in H$. We can then easily show that

$$[\gamma_1, \gamma_2]_D = [\gamma_2, \gamma_1]_D \qquad ...(4.5)$$

thus establishing that D is symmetric. Further, the potentials $W_1 = W[\gamma_1]$ and $W_2 = W[\gamma_2]$ are harmonic in S, and on S, by virtue of Green's identity,

$$\int_S W_{1+} \frac{\partial W_2}{\partial n_+} dS = \int_S W_{2+} \frac{\partial W_1}{\partial n_+} dS,$$

which gives, with the help of (2.3) and (4.5)

$$[K^*\gamma_1, \gamma_2]_D = [K^*\gamma_2, \gamma_1]_D. \quad \ldots(4.6)$$

Hence, we have established the following theorem :

Theorem 4 — K and K^* are symmetric with respect to G-norm and D-norm respectively.

In particular, if μ_i and μ_j are eigen-functions of K corresponding to λ_i^{-1} and λ_j^{-1} and $\gamma_i = G\mu_i$, then

$$\left.\begin{aligned} &[\mu_i, \mu_j]_G = \delta_{ij} \\ \text{and} \quad &[\gamma_i, \gamma_j]_D = \left(\frac{1}{\lambda_i \lambda_j} - 1\right)\delta_{ij} \end{aligned}\right\} \quad \ldots(4.7)$$

where μ_i are normalized with respect to G-norm, for $i = j$, $\| \mu_i \|_G = 1$ and

$$\| \gamma_i \|_D^2 = \left(\frac{1}{\lambda_i^2} - 1\right), \; \| \gamma_i \|_D = \frac{\sqrt{1 - \lambda_i^2}}{\lambda_i} = C_i \quad \ldots(4.8)$$

say. Thus, scale-factors $1/C_i$ will appear when normalization with respect to D-norm is carried out.

Let $\| K \|_G$ denote the norm of K with respect to G-norm, then as in Nayar (1974),

$$\| K \|_G \leqslant \frac{1}{\lambda_1}.$$

Using the result of Riesz and Nagy (1955) regarding symmetric transformations

$$\sup \|K\mu \|_G = \| K \|_G \| \mu \|_G$$

and by (4.3), we can show that

$$\| \gamma \|_D \leqslant C_1 \| \mu \|_G. \quad \ldots(4.9)$$

Hence, we have the following result :

Theorem 5 — The G-norm and D-norm are linearly comparable and in particular $\| \gamma \|_D \leqslant C_1 \| \mu \|_G$, where C_1 is a definite constant depending on the eigen-value λ_1^{-1}.

As done by Nayar (1974), we may analogously define the norm of the adjoint transformation K^* with respect to the D-norm as

$$\| K^* \|_D = \sup \frac{[K^*\gamma, \gamma]_D}{[\gamma, \gamma]_D}, \gamma = G\mu \in H. \quad \text{...(4.10)}$$

Using (4.3), we can easily show that

$$[K^*\gamma, \gamma]_D = \| K \|_G [\gamma, \gamma]_D. \quad \text{...(4.11)}$$

Therefore,

$$\| K^* \|_D = \| K \|_G = \frac{1}{\lambda_1}. \quad \text{...(4.12)}$$

Since K and K^* are symmetric, it follows that (Riesz and Nagy 1955)

$$\left.\begin{array}{l} \| K\mu \|_G \leqslant \lambda_1^{-1} \| \mu \|_G \\ \text{and} \\ \| K^*\gamma \|_D \leqslant \lambda_1^{-1} \| \gamma \|_D \leqslant C_1 \lambda_1^{-1} \| \mu \|_G \leqslant \lambda_1^{-2} \| \mu \|_G \end{array}\right\} \quad \text{...(4.13)}$$

since $C_1 < \lambda_1^{-1}$. Thus, we have the following result.

Theorem 6 — The transformations K and K^* are bounded in norm, i.e.. $\| K^* \|_D = \| K \|_G = \lambda_1^{-1}$. Moreover,

$$\left.\begin{array}{l} \| K\mu \|_G \leqslant \lambda_1^{-1} \| \mu \|_G \\ \text{and} \\ \| K^*\gamma \|_D \leqslant \lambda_1^{-2} \| \mu \|_G. \end{array}\right\} \quad \text{...(4.14)}$$

We have now the requisite analytical information to discuss in detail the set of transformations DG and GD, yielded by G and D, the left symmetrizers of K and K^*, respectively.

5. The Transformations DG and GD

It is known that K and K^2 are continuous in G-norm (Nayar 1974) and the identity transformation I is always continuous. Thus, it is evident that $DG = K^2 - I$ is continuous in G-norm. Moreover, in view of (4.9), convergence in G-norm implies convergence in D-norm. Therefore, DG is continuous with respect to both the norms. However, a rigorous proof following Nayar (1974) has been developed (Nayar 1975), wherein it has been shown that G transforms weakly (strongly) convergent sequences into weakly (strongly) convergent sequences in G-norm as well as in D-norm. The same applies to the transformation DG, since strong convergence implies weak convergence. Hence, the following result is obtained.

Theorem 7 — The transformation DG is continuous with respect to G-norm as well as D-norm.

Similar arguments may be applied to the transformation $GD = K^{*2} - I$ in which the kernel of the adjoint problem K^* occurs. We have earlier stated (Section 4) that K^* is symmetric in the space endowed with D-norm. The following may easily be shown.

Theorem 8 — The transformations K^*, K^{*2} and GD are continuous with respect to D-norm.

Now, from this vantage point, we look at the transformations (2.6) and (2.7) in the form

$$I = K^2 - DG; \; I = K^{*2} - GD. \qquad \ldots(5.1)$$

We call the right hand members above, reproducing kernels, the left hand members being the identity transformations. The continuity of various constituent transformations in (5.1) with respect to the two norms has been established in the preceding theorems. We may now summarize the above results as follows.

Theorem 9 — If K and K^* be the kernels of the integral equations associated with the induced potential problem and its adjoint, G and D, their left symmetrizers, then the transformations $K^2 - DG$ and $K^{*2} - GD$ are continuous with respect to G-norm and D-norm respectively. They are also the reproducing kernels in spaces endowed with such norms.

6. Generalized Fourier Expansions

The orthogonal relations (4.7), under the two norms, give generalized Fourier expansions analogous to those noted in Nayar (1974).

Let

$$f_i = [f, \mu_i]_G \text{ and } f_i' = [f, \gamma_i]_D; \; f \in H \qquad \ldots(6.1)$$

denote generalized Fourier coefficients with respect to G-norm and D-norm, respectively. Evidently, by virtue of (4.8),

$$\gamma_i' = [\gamma_i, \gamma_i]_D = \| \gamma_i \|_D^2 = C_i^2 = \frac{1 - \lambda_i^2}{\lambda_i^2}. \qquad \ldots(6.2)$$

Thus, in the space endowed with D-norm as well, linear, bilinear and quadratic expansions hold.

Let

$$f = \Sigma\, \alpha'_i\, \gamma_i,$$

where α'_i are generalized Fourier coefficients,

By virtue of biorthogonal property $(\mu_i, \gamma_j) = \delta_{ij}$, Theorem 1 and (6.2), we have

$$\alpha'_i = (f, \mu_i) = \left(f, \frac{\lambda_i^2}{1-\lambda_i^2} D\gamma_i\right) = \frac{1}{C_i^2}[f, \gamma_i]_D = \frac{f'_i}{\gamma'_i}$$

Therefore,

$$f = \sum \frac{f'_i}{\gamma'_i} \gamma_i \qquad \text{...(6.3)}$$

whence

$$K^* f = \sum \lambda_i^{-1} \cdot \frac{f'_i}{\gamma'_i} \gamma_i \qquad \text{...(6.4)}$$

and

$$Df = \sum \frac{f'_i}{\gamma'_i} D\gamma_i = \sum f'_i\, \mu_i. \qquad \text{...(6.5)}$$

Moreover, the following results can be obtained easily:

$$[f, f]_D = \sum \frac{f'^2_i}{C_i^2} \qquad \text{...(6.6)}$$

and

$$[K^* f, f]_D = \sum \lambda_i^{-1} \cdot \frac{f'^2_i}{C_i^2} \cdot \qquad \text{...(6.7)}$$

Since λ_i^{-1} is an ordered set with $|\lambda_i^{-1}| < 1$ and $\lambda_i^{-1} \to 0$, we have from (6.6) and (6.7),

$$[K^* f, f]_D < [f, f]_D \qquad \text{...(6.8)}$$

which is analogous to the result noted for the transformation K in the space H endowed with G-norm

$$[Kf, f]_G < [f, f]_G, \quad f \in H.$$

Likewise, we may obtain the following Hilberts' bilinear expansion.

Let f, $g \in H$, endowed with D-norm. Then

$$[K^*g, f]_D = (K^*g, Df) = \left(\sum \frac{g_i'}{\lambda_i \gamma_i'} \gamma_i, \sum f_i' \mu_i\right) = \sum \frac{g_i' f_i'}{\lambda_i \gamma_i'}. \quad \ldots(6.9)$$

By virtue of the symmetric property of K^* with respect to D-norm, we have

$$[K^*g, \gamma_i]_D = [g, K^*\gamma_i]_D = \lambda_i^{-1} g_i'.$$

Thus,

$$[K^*g, f]_D = \sum \frac{[K^*g, \gamma_i]_D [f, \gamma_i]_D}{[\gamma_i, \gamma_i]_D} \quad \ldots(6.10)$$

which is analogous to the expansion noted in Nayar (1974)

$$[Kg, f]_G = \Sigma [Kg, \mu_i]_G [f, \mu_i]_G.$$

The scale factors $\frac{1}{\gamma_i'}$ appear in (6.10) because μ_i are normalized with respect to G-norm and γ_i are not normalized with respect to D-norm. Proof for the above expansions may be developed on identical lines as in Nayar (1974) employing Bessel's inequality and the maximum principle. The above results may be summarized as follows.

Theorem 10 — If f, $g \in H$, endowed with D-norm in which K^* with spectrum $\{\lambda_i^{-1}, \gamma_i\}$ is symmetric and continuous and if $f_i' = [f, \gamma_i]_D$, then the following expansions hold:

(i) $f = \sum \frac{f_i'}{\gamma_i'} \gamma_i$

(ii) $K^*f = \sum \lambda_i^{-1} \frac{f_i'}{\gamma_i'} \gamma_i$

(iii) $Df = \Sigma f_i' \mu_i$

(iv) $[f, f]_D = \sum \frac{f_i'^2}{\gamma_i'}$

(v) $[K^*f, f]_D = \sum \frac{\lambda_i^{-1}}{\gamma_i'} f_i'^2$

(vi) $[K^*g, f]_D = \sum \frac{\lambda_i^{-1}}{\gamma_i'} f_i' g_i' = \sum \frac{[K^*g, \gamma_i]_D\,[f, \gamma_i]_D}{[\gamma_i, \gamma_i]_D}.$ ...(6.11)

In order to establish the convergence of some of the above series and other results analogous to those proved earlier by Nayar (1974), we recall from Howland and Vaillancourt (1966), the following convergent series:

$$\sum \frac{V_i^2}{\lambda_i^2} \leqslant B^2; \quad V_i = G\mu_i = \gamma_i. \qquad \text{...(6.12)}$$

This is evidently equivalent to

$$\sum (K^*\gamma_i)^2 = \sum \frac{\gamma_i^2}{\lambda_i^2} \leqslant B^2. \qquad \text{...(6.13)}$$

Let $f \in H$ be bounded in D-norm, then if

$$f = \sum a_i \gamma_i, \; a_i = \frac{f_i'}{\gamma_i'} = (f, \mu_i)$$

is such that $\sum a_i^2$ is finite, i.e., $\sum \left(\frac{f_i'}{\gamma_i'}\right)^2$ converges absolutely. Further, let

$$K_n^* f = \sum_{i=1}^{n} \frac{f_i'}{\gamma_i'} \cdot \frac{\gamma_i}{\lambda_i}. \qquad \text{...(6.14)}$$

Using Bessel's inequality and (6.13), we have

$$K_n^* f \leqslant \left(\sum_{i=1}^{n} \frac{f_i'^2}{\gamma_i'^2}\right)^{1/2} \left(\sum_{i=1}^{n} \frac{\gamma_i^2}{\lambda_i^2}\right)^{1/2} \leqslant B \, \| f \|_D. \qquad \text{...(6.15)}$$

Also,

$$K_n^* f = \sum_{i=1}^{n} \frac{(f, \mu_i)\, \gamma_i}{\lambda_i}$$

Thence,

$$K_n^* = \sum_{i=1}^{n} \frac{\gamma_i \mu_i}{\lambda_i}$$

gives the degenerate form of K^*, the nth linear approximation, which is also bounded in norm and tends to it in D-norm, i.e., the following results can be established on identical lines (Nayar 1974),

$$\| K_n^* \|_D < \frac{1}{\lambda_1}$$

$$\| K_n^* \|_D^2 < \sqrt{T_4}$$

where T_4 is the trace of the 4th iterate, K^4, known to be finite and

$$\| K^* - K_n^* \|_D < \frac{1}{\lambda_{n+1}} \to 0. \qquad \text{...(6.16)}$$

Thus, the series

$$\sum \frac{f_i'^2}{\gamma_i'} \frac{1}{\lambda_i} \text{ and } \sum \frac{f_i'^2}{\gamma_i'} \cdot \frac{1}{\lambda_i^2}$$

converge absolutely. They are equivalent to $\sum \frac{\lambda}{1 - \lambda_i^2} f_i'^2$

and $\sum \frac{1}{1 - \lambda_i^2} f_i'^2$ respectively. Further, $\sum f_i'^2 = \sum a_i^2 \gamma_i'^2 < \sum a_i^2$,

which is finite. Thus, $\sum f_i'^2$ also converges absolutely. Using the relations (3.1) - (3.5), we have

$$\sum \frac{f_i'^2}{1 - \lambda_i^2} = \sum \left(f, \frac{\partial V_i}{\partial n_-} \right) \left(f, \frac{\partial V_i}{\partial n_+} \right) \Big/ \lambda_i^2 \qquad \text{...(6.17)}$$

and

$$\sum \frac{V_i^2}{\lambda_i^2} = \sum \frac{W_{i+} W_{i-}}{1 - \lambda_i^2} \leqslant B^2. \qquad \text{...(6.18)}$$

Since $\frac{1}{\lambda_i^2} < \frac{1}{1 - \lambda_i^2}$, the above is the majorant for the series $\sum \frac{W_{i+} W_{i-}}{\lambda_i^2}$, which converges absolutely. Thus, we have proved the following theorem.

Theorem 11 — If $f \in H$ and V_i and W_i are the Poincaré's fundamental functions, then on S, the series

$$\sum \left(f, \frac{\partial V_i}{\partial n_+}\right)\left(f, \frac{\partial V_i}{\partial n_-}\right)\Big/\lambda_i^2 \qquad \text{...(6.19)}$$

and

$$\sum \frac{(W_{i+})\,(W_{i-})}{\lambda_i^2} \qquad \text{...(6.20)}$$

converge absolutely.

7. Solution of the Adjoint Integral Equation

We are now in a position to give the formal solution of (2.4) in terms of the eigen set of K^*, as was done for (1.1) by Nayar (1974) in the form of a Mittag-Leffler type series.

Theorem 12 — The functional equation

$$\gamma = g + \lambda K^* \gamma \quad \text{[eqn. (2.4)]}$$

has the solution

$$\gamma = g + \lambda \sum \frac{(g, \mu_i)}{\lambda_i - \lambda} \gamma_i, \;\; \lambda \neq \lambda_i \qquad \text{...(7.1)}$$

where $G\mu_i = \gamma_i$ are the eigen-functions of K^* corresponding to non-zero eigen-values λ_i^{-1} and g is the boundary value of the potential of double layer of moment γ on S.

Let $\quad \gamma = \Sigma\, \alpha_i \gamma_i \qquad \text{...(7.2)}$

where α_i are generalized Fourier coefficients with respect to orthogonal set $(\mu_i, \gamma_j) = \delta_{ij}$. Substituting (7.2) in (2.4) and taking scalar product with μ_i, we get,

$$\alpha_i = \frac{\lambda_i}{\lambda_i - \lambda} (g, \mu_i)$$

whence

$$\gamma = \sum \frac{\lambda_i}{\lambda_i - \lambda} (g, \mu_i)\, \gamma_i. \qquad \text{...(7.3)}$$

This can be summed partially, since

$$g = \Sigma (g, \mu_i) \gamma_i.$$

Thus, (7.3) becomes

$$\gamma = g + \lambda \sum \frac{(g, \mu_i)}{\lambda_i - \lambda} \gamma_i. \quad \ldots(7.4)$$

The harmonic function sought is then the potential due to the double layer γ.

$$W[\gamma] = W[g] + \lambda \sum \frac{(g, \mu_i)}{\lambda_i - \lambda} W_i \quad \ldots(7.5)$$

where $W_i = W[\gamma_i]$ are Poincaré's fundamental functions. The normal derivative of the potential will be given by

$$D\gamma = Dg + \lambda \sum \frac{(g, \mu_i)}{\lambda_i - \lambda} D\gamma_i.$$

Since $\mu_i \sim D\gamma_i$, we have $(g, \mu_i) D\gamma_i = g_i' \mu_i$

$$\therefore D\gamma = Dg + \lambda \sum \frac{g_i'}{\lambda_i - \lambda} \mu_i. \quad \ldots(7.6)$$

Further,

$$(g, \mu_i) = \left(g, \frac{D\gamma_i}{C_i^2}\right) = \frac{g_i'}{C_i^2}.$$

Thence

$$\gamma = g + \lambda \sum \frac{g_i'}{\lambda - \lambda} \cdot \frac{\gamma_i}{C_i^2}. \quad \ldots(7.7)$$

For comparison, we recall the solution of the induced potential problem

$$\mu = f + \lambda K \mu : f = \lambda \frac{\partial X}{\partial n} \quad \ldots(1.1)$$

which in the present notation becomes

$$G\mu = Gf + \lambda \sum \frac{(f, \gamma_i)}{\lambda_i - \lambda} \gamma_i, \quad \gamma_i = G\mu_i \quad \ldots(7.8)$$

Since $g_i' = (Dg, G\mu_i)$, we see that if the spectrum of K, i.e., $\{\lambda_i^{-1}, \mu_i\}$ is known, then the solutions of both (1.1) and (2.4) are respectively given by (7.8) and (7.4). Alternatively, if the spectrum of K^*, i.e., $\{\lambda_i^{-1}, \gamma_i\}$ is known, then these solutions are given by (7.8) and (7.7) respectively. Thus, the spectrum of either kernel solves both the functional equations.

REFERENCES

Blumenfeld, J., and Mayer, W. (1914). Über Poincaré sche fundamental funktionen. *Sber. Akad. Wiss. Wien*, **123**, 2011-47.

Howland, J. L. (1968). Symmetrizing kernels and the integral equations of first kind of classical potential theory. *Proc. Am. math. Soc.*, **19**, 1-7.

Howland, J. L., and Vaillancourt, R. (1966). Series expansions of induced potentials. *Math. Anal. Applic.*, **16**, (3), 385-95.

Kellogg, O. D. (1929). Foundations of Potential Theory. Ungar, New York.

Nayar, B. M. (1974). Induced potential problem in three dimensions—I (Properties of the associated kernel). *Presented at the 40th Annual Conference of the Indian Mathematical Society.*

———— (1975). Induced potential problem in three dimensions—II (Properties of the adjoint kernel). *Presented at the 41st Annual Conference of the Indian Mathematical Society.*

Riesz, F., and Nagy, B. SZ. (1955). Functional Analysis, translated from the second French edition of Lecons d' Analyse Fonctionelle. Ungar, New York.

Tauber, A. (1897). Über das potential einer Doppelbelegung. *Mh. Math. Phys.*, **8**, 79-86.

———— (1898). Über Einige Sätze der potential theorie. *Mh. Math. Phys.*, **9**, 74-88.