# AN EXTENSION OF THE FIXED POINT PROBABILITY VECTOR FOR THE SUCCESSIVE COMPOSITION OF TRANSITION MATRICES

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In this paper we have tried to establish a fixed point probability vector for the successive composition of a finite set of transition matrices—an extension of Heenan's result (1970). An example is given to show that this condition fails in the case of a finite product of such like matrices

The well-known Banach contraction principle has been used in the field of fixed points in various forms. Browder (1966) applied this principle for various types of operators, linear or non-linear, acting on various spaces to evaluate the fixed point property. Belluce and Kirk (1969) used it for non-expansive mappings in Banach space. Heenan (1970) devised a method to determine the fixed point in a transition matrix. He showed that if $A$ is the transpose of a probability matrix $W$, then $\exists\ w \in W$, such that $A(w) = w$. By a transpose of a probability matrix $W$, we mean a matrix in which the sum of each column is unity.

The object of the present study is to extend the above result for the set of non-singular transition matrices which reduces Heenan's result (1970) as a corollary. Examples are cited to illustrate the above fact and its failure. Throughout the paper, $W$ stands for the set of matrices under composition and $A_i$'s denote the transpose of the corresponding transition matrices.

§1. The following definition of Markov system will be required to explain the concept of transition matrix.

*Definition* 1.1 — Let us consider a probabilistic system which can assume any one of the finite number of states, say $N$. These states are all distinct and identifiable and thus can be ordered by an index $k$, where $k = 1, 2, ..., N$. At the start of a certain prescribed interval, the system may or may not change from its present state to one of the other $N - 1$ possible states. The start of these intervals at which the transitions take place need not be identifiable with equal time intervals. There exists a certain class of stochastic processes for which the transition probability

$p_{ij}$ for the system to change from state $i$ to state $j$ at the beginning of one of these intervals is constant and depends only upon $i$ and $j$. In other words, the transition probability is independent of the past history of the system, equivalently, any past transitions. Such a stochastic model is called a simple Markov system.

We now define the transition matrix.

*Definition* 1.2 — The transition matrix, which is also known as a stochastic matrix, is the mathematical representation of Markov system where each element in the matrix $p_{ij}$ is the probability that a system now in state $i$ will transit to state $j$ at the beginning of the next interval. Thus, the matrix becomes

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{bmatrix} \qquad \ldots(1.1)$$

where $\quad 0 \leqslant p_{ij} \leqslant 1; \quad i, j = 1, 2, \ldots, N$

and $\quad \sum_{i=1}^{N} p_{ij} = 1; \qquad i = 1, 2, \ldots, N.$

The terms $p_{ii}$ denote the possibilities for the system to remain at the same state from one interval to another.

If we define the probability that a Markov system is in state $i$ during the $n$th period as $P_i(n)$, then

$$P_j(n+1) = \sum_{i=1}^{N} p_{ij} P_i(n), \quad j = 1, 2, \ldots, N. \qquad \ldots(1.2)$$

There are two properties that a simple Markov system possesses in most of the cases :

(1) The limiting value of $P_i(n)$ $(i = 1, 2, \ldots, N)$, as $n \to \infty$ is unique. These values are called steady-state probabilities of the Markov process.

(2) The limiting value of a property is independent of the state from which the system initially starts at $n = 0$.

§2. In this section, we proceed to prove the following theorem.

*Theorem* 2.1 — Let $A_1, A_2, A_3, \ldots, A_n$ be the finite set of non-singular matrices and $W$ be the set of corresponding probability matrices, then $\exists\, w \in W$, such that

$$A_1 \circ A_2 \circ \ldots \circ A_n(w) = w. \qquad \ldots(2.1)$$

PROOF : To prove (2.1), we shall employ the technique of mathematical induction.

The theorem is obviously true, when $n = 1$.

Now supposing that theorem (2.1) is true for $n$, we shall show that it is also true for $(n+1)$. We have

$$A_1 \circ A_2 \circ A_3 \circ \ldots \circ A_{n+1}(w) = A_1 \circ (A_2 \circ A_3 \circ \ldots \circ A_{n+1})(w). \qquad \ldots(2.2)$$

Using the property that every non-singular matrix $A$ can be reduced to the form $PQ^{-1}$, where $P$ is symmetrical and $Q$ is diagonal, each element of $A_i$'s can be written in the following ratio form :

$$a^1_{ij} = \frac{p^1_{ij}}{q^1_i},\ a^2_{ij} = \frac{p^2_{ij}}{q^2_i},\ \ldots,\ a^n_{ij} = \frac{p^n_{ij}}{q^n_i} \qquad \ldots(2.3)$$

where $q^k_i$'s $\in Q$, $k = 1, 2, \ldots, n$,

$i, j = 1, 2, \ldots, N$.

At the steady state, the components of vector $w$ are given by

$$w^k_i = \frac{q^k_i}{\sum\limits_{j=0}^{N-1} q^k_j},\ k = 1, 2, \ldots, n;\ \ i = 0, 1, 2, \ldots, N-1. \qquad \ldots(2.4)$$

In view of relations (2.3) and (2.4), we get

$$A_1 \circ (A_2 \circ A_3 \circ \ldots \circ A_{n+1})(w) =$$

$$\sum_{i,j=0}^{N-1} \frac{p^1_{ij}}{q^1_i} w_i \circ \sum_{i,j=0}^{N-1} \frac{p^k_{ij}}{q^k_i} w^k_i \qquad (k = 2, 3, \ldots, (n+1))$$

$$= \sum_{i,j=0}^{N-1} \frac{p^{k+1}_{ij}}{q^{k+1}_i} w^{k+1}_i \qquad (k = 1, 2, \ldots, n)$$

$$= w.$$

(Here the $q_i$'s common to each column will be identical, because all $A_i$'s are non-singular matrices of the set of corresponding probability matrices $W$).

This completes the proof of the theorem.

The components of the probability vector $w$ in terms of $p^k_{ij}$ and $q^k_j$ can be obtained from the following relations.

$$\sum_{i=0}^{N-1} \frac{p^k_{ij}}{q^k_i} w^k_i = w^k_j$$

where $k = 1, 2, \ldots, n; \quad j = 0, 1, 2, \ldots, N-1.$ ...(2.5)

Further, if

$$w^k_j = \frac{q^k_j}{\sum_{i=0}^{N-1} q^k_i} \qquad i, j = 0, 1, 2, \ldots, N-1 \quad k = 1, 2, \ldots, n,$$

then (2.5) becomes

$$\sum_{i=0}^{N-1} p^k_{ij} = q^k_j \qquad (k = 1, 2, \ldots, n) \qquad \ldots(2.6)$$

which is valid if the matrices $p_i$'s are symmetric.

*Corollary* — If $A_1 = A_2 = A_3 = \ldots = A_n$ be the finite set of non-singular matrices, and $W$ be the set of corresponding ($w_1 = w_2 = \ldots = w_n = W$) probability matrices, then $\exists\, w \in W$, such that

$$A^n w = w. \qquad \ldots(2.7)$$

**Illustrations**

*Example* 1 — To prove $A_1 \circ A_2(w) = w$, where

$$A_1 = \begin{bmatrix} 0.7 & 0.2 & 0.1333 \\ 0.1 & 0.6 & 0.0667 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.67 & 0.22 & 0.147 \\ 0.11 & 0.56 & 0.073 \\ 0.22 & 0.22 & 0.78 \end{bmatrix}$$

They can be reduced to the form $p^i\, Q^{i-1}$, where $i = 1, 2, \ldots$

$$A_1 = \begin{bmatrix} 7 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 12 \end{bmatrix} \begin{bmatrix} 1/10 & & \\ & 1/5 & \\ & & 1/15 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 67 & 11 & 22 \\ 11 & 28 & 11 \\ 22 & 11 & 117 \end{bmatrix} \begin{bmatrix} 1/100 & & \\ & 1/50 & \\ & & 1/150 \end{bmatrix}$$

Here, $q_0^1 = 10,\ q_1^1 = 5,\ q_2^1 = 15, \quad \therefore\ \Sigma q^1 = 30$ and

$$w_0^1 = 10/30 = 1/3,\ w_1^1 = 5/30 = 1/6,\ w_2^1 = 15/30 = 1/2. \text{ And}$$

$$q_0^2 = 100,\ q_1^2 = 50,\ q_2^2 = 150, \quad \therefore\ \Sigma q^2 = 300,$$

$$w_0^2 = 100/300 = 1/3,\ w_1^2 = 50/300 = 1/6,\ w_2^2 = 150/300 = 1/2.$$

Hence,

$$A_1 \circ A_2(w) = w.$$

The physical interpretation of $A_1 \circ A_2$ is as follows. Out of the total 30 and 300 events respectively, the process in the state 0, state 1 and state 2, is 10, 5 and 15 times for $A_1$ and 100, 50 and 150 times for $A_2$ respectively.

The theorem is not valid if we replace the composition of the matrices by the finite product of matrices, as is obvious from the following example.

*Example* 2 — To prove $A_1 \times A_2(w) \neq w$, consider $A_1$, $A_2$ as given in Example 1. We have

$$A_1 \times A_2 = \begin{bmatrix} 0.7 & 0.2 & 0.1333 \\ 0.1 & 0.6 & 0.0667 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \times \begin{bmatrix} 0.67 & 0.22 & 0.147 \\ 0.11 & 0.56 & 0.073 \\ 0.22 & 0.22 & 0.78 \end{bmatrix}$$

which on simplification reduces to

$$\begin{bmatrix} 0.520326 & 0.295326 & 0.221474 \\ 0.147647 & 0.372674 & 0.110526 \\ 0.332 & 0.332 & 0.9326 \end{bmatrix}$$

Here, $P$ is no longer symmetric. Thus,

$$A_1 \times A_2(w) \neq w.$$

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