AN OPEN DISK THEOREM AND ITS CONTRIBUTIONS TO HYDRODYNAMIC STABILITY

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An open disk theorem which gives an upper bound of the eigenvalues for a certain class of problems defined by a coupled system of linear homogeneous partial differential equations with appropriate homogeneous boundary conditions in a simply connected subset V of the n-dimensional Euclidean space R^n is established. The reduced form of the theorem in one dimension is then shown to lead to results, some of which are still not known, from four major areas of theoretical investigation in the field of hydrodynamic stability.

INTRODUCTION

Consider the coupled system of $2n$ linear homogeneous partial differential equations given by

$$l[\nabla^2 - m^2][A\nabla^2 - m^2B - lC]\,\psi = \lambda^2 lN\Phi - m^2F(x)\,\psi \qquad \ldots(1)$$

and

$$[E\nabla^2 - m^2G - lH(x)]\,\Phi = -N\psi \qquad \ldots(2)$$

which is valid in a simply connected open subset V of R^n together with homogeneous boundary conditions

$$\left.\begin{array}{ll} & \psi_i = 0 = \Phi_i \\ \text{and either} & \dfrac{\partial\psi_i}{\partial n} = 0 \\ \text{or} & \nabla^2\psi_i = 0 \end{array}\right\} \qquad \ldots(3)$$

$i = 1, 2, 3, \ldots, n$, on the boundary S of V. In the above equations, A, B, C, N, E and G are $n \times n$ Hermitian matrices whose entries, in general, are complex constants with C being non-singular and positive definite, $F(x)$ and $H(x)$, x representing a point of V, i.e., $x = (x_1, x_2, \ldots, x_n)$, are $n \times n$ Hermitian matrices with entries as continuous complex valued functions at each point x in V with $H(x)$ being positive definite everywhere in V; m^2 and λ^2 are positive constants, ∇^2 stands for the n-dimensional

Laplacian operator $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ and $\frac{\partial}{\partial n}$ denotes the differentiation along the outward drawn normal at any point on the boundary of V; ψ and Φ are $n \times 1$ matrices with entries ψ_i and Φ_i respectively, which are complex valued functions on V, while l is a constant which can, in general, be complex. Further, $\nabla^k\psi$ $[k = 2, 4]$ denotes the matrix $[\nabla^k\psi_i]_{n\times 1}$ and similarly $\nabla^2\Phi$ stands for $[\nabla^2\Phi_i]_{n\times 1}$.

Equations (1) and (2) together with boundary conditions (3) define a generalized eigenvalue problem in the sense that one is interested in finding out, for given values of m^2, λ^2, A, B, C, N, E, F, G and H, those values of l for which non-trivial ψ and Φ satisfying eqns. (1), (2) and (3) exist. In the subsequent discussions, we shall call such values of l, ψ and Φ as solutions of the problem and denote such a solution by the symbol (l, ψ, Φ).

The main result established in the paper is that if $(l = l_r + il_i, \psi, \Phi)$ is a solution of the problem with $l_i \neq 0$, then

$$|l|^2 < \frac{f_1}{C_1} \qquad \ldots(4)$$

where l_r and l_i respectively denote the real and imaginary parts of l; f_1 is the largest eigenvalue of $F(x)$ in V and C_1 is the smallest eigenvalue of C. The positive definite and non-singular character of the Hermitian matrix ensures that $C_1 > 0$.

The above result is then shown to lead to the following consequences in certain major areas of theoretical investigation in the field of hydrodynamic stability :

(i) An upper and a lower bound for the frequency of oscillations of the marginal modes in the linear axisymmetric stability problem of a narrow gap viscous Couette flow with an axial pressure gradient can be explicitly given in terms of the Reynolds number and the concerned wave number and further that non-oscillatory modes cannot exist for wave numbers exceeding $\sqrt{12}$ and, therefore, in particular, the principle of exchange of stabilities is not valid for the problem.

The above results are completely in accord with the numerical calculations of Chandrasekhar (1960, 1961) for the problem, though Chandrasekhar does not give an explicit mathematical proof to the effect that the principle of exchange of stabilities is not satisfied. His calculations are thus based on the assumption of an overstable marginal state.

(ii) An upper and a lower bound for the frequency of oscillations of the marginal modes, in the linear stability problem of thermohaline convection for free as well as rigid boundaries under the assumption that the diffusion of salt is much slower than the diffusion of heat, can be explicitly given in terms of the salinity Rayleigh number and the thermal Prandtl number.

The above results are completely in accord with the exact solution of Veronis (1965) for the problem. However, the solution of Veronis suffers from the defect that it is valid for the case of free boundaries only. The establishment of bounds for the case of rigid boundaries also, especially when the calculations in this case get very complicated, is a new contribution to this field of enquiry.

(iii) An arbitrary mode, whether stable, marginal or unstable, in the linear stability problem of thermal convection, is necessarily non-oscillatory (Pellew and Southwell 1940).

(iv) The complex growth rate n ($= n_r + in_i$, n_r and n_i being real constants) of an arbitrary oscillatory mode, whether stable, marginal or unstable, in the linear stability problem of the Rayleigh-Taylor model of a Boussinesq fluid with constant coefficient of viscosity, must lie in an open disk in the $n_r n_i$ plane with centre as the origin and radius as

$$\left[\max_{\text{Flow Domain}} \left\{-\frac{gD\rho}{\rho_0}\right\}\right]^{1/2}$$

where g stands for the acceleration due to gravity; ρ, the density, $\rho_0 > 0$ is a constant whose dimensions are that of density and $D\rho = \frac{d\rho}{dz}$, z being the vertical coordinate (Banerjee and Kalthia 1971).

Remark 1 : To the best of our knowledge, the problem of the existence of a solution to eqns. (1), (2) and (3) has not been attempted mathematically in its full generality. However, there do exist quite a few physical problems in the field of hydrodynamic stability whose governing equations coincide exactly with eqns. (1), (2) and (3) for the special case of two dimensions, when the matrices and numbers involved have some special characters. And further, these specialized equations have solutions which can be obtained by further reducing them to one dimension (Chandrasekhar 1961).

2. Mathematical Analysis

We prove the following theorems.

Theorem 1 — If (l, ψ, Φ) is a solution of eqns. (1), (2) and (3) with $l_i \neq 0$, then

$$|l|^2 < \frac{f_1}{C_1}.$$

Proof : Since $l_i \neq 0$, we can write eqn. (1) as

$$[\nabla^2 - m^2][A\nabla^2 - m^2B - lC]\psi = \lambda^2 N\Phi - \frac{m^2}{l}F(x)\psi. \qquad \ldots(5)$$

Further, let $\psi^\dagger$ denote the transpose conjugate of ψ.

Then left multiplying eqn. (5) by $\psi^\dagger$ and integrating the resulting equation over the domain V, we get

$$\int_V \psi^\dagger [A \nabla^4 - \{m^2 A + m^2 B + lC\} \nabla^2 + m^4 B + lm^2 C] \psi \, dV$$

$$= \lambda^2 \int_V \psi^\dagger N\phi \, dV - \frac{m^2}{l} \int_V \psi^\dagger F(x) \psi \, dV. \qquad \ldots(6)$$

The integral $\int_V \psi^\dagger A \nabla^4 \psi \, dV$ occurring in eqn. (6) can be written as

$$\int_V \psi^\dagger A \nabla^4 \psi \, dV = \int_V \psi_i^* A_{ij} \nabla^4 \psi_j \, dV = A_{ij} \int_V \psi_i^* \nabla^2(\nabla^2 \psi_j) \, dV \qquad \ldots(7)$$

where ψ_i^* is the complex conjugate of ψ_i, (A_{ij}) is the matrix A and the summation convection is applied for the repeated suffixes which take integral values 1, 2, 3, ..., n.

Now making repeated use of Gauss' theorem in n-dimension and boundary conditions (3), we obtain

$$\int_V \psi_i^* \nabla^2(\nabla^2 \psi_j) \, dV = \int_S \psi_i^* \frac{\partial}{\partial n} (\nabla^2 \psi_j) \, ds - \int_V \vec{\nabla} \psi_i^* \cdot \vec{\nabla}(\nabla^2 \psi_j) \, dV$$

$$= - \int_V \vec{\nabla} \psi_i^* \cdot \vec{\nabla}(\nabla^2 \psi_j) \, dV$$

$$= - \int_S \nabla^2 \psi_j \frac{\partial \psi_i^*}{\partial n} dS + \int_V \nabla^2 \psi_j \nabla^2 \psi_i^* \, dV$$

$$= \int_V \nabla^2 \psi_i^* \nabla^2 \psi_j \, dV \qquad \ldots(8)$$

where $\vec{\nabla}$ stands for the usual gradient operator in n-dimensions, while the symbol (.) is the dot product between two vectors

From eqns. (7) and (8), we have

$$\int_V \psi^\dagger A \nabla^4 \psi \, dV = A_{ij} \int_V \nabla^2 \psi_i^* \nabla^2 \psi_j \, dV = \int_V \nabla^2 \psi^\dagger A \nabla^2 \psi \, dV. \qquad \ldots(9)$$

Also,

$$\int_V \psi^\dagger A \nabla^2 \psi \, dV = A_{ij} \int_V \psi_i^* \nabla^2 \psi_j \, dV$$

$$= A_{ij} \left[\int_S \psi_i^* \frac{\partial \psi_j}{\partial n} dS - \int_V \vec{\nabla} \psi_i^* \cdot \vec{\nabla} \psi_j \, dV \right]$$

(*equation continued on p.* 146)

$$= - A_{ij} \int_V \vec{\nabla} \psi_i^* \cdot \vec{\nabla} \psi_j dV$$

$$= - \int_V \frac{\partial \psi_i^*}{\partial x_r} A_{ij} \frac{\partial \psi_j}{\partial x_r} dV$$

$$= - \int_V \frac{\partial \psi^\dagger}{\partial x_r} A \frac{\partial \psi}{\partial x_r} dV \qquad \text{...(10)}$$

where $r = 1, 2, 3, \ldots, n$ and $\frac{\partial \psi}{\partial x_r}$ represents the matrix $\left[\frac{\partial \psi_i}{\partial x_r}\right]_{n \times 1}$, $i = 1, 2, 3, \ldots, n$.
Now making use of relations (9) and (10), we have from eqn. (6)

$$\int_V \{\nabla^2 \psi^\dagger A \nabla^2 \psi + \nabla_r \psi^\dagger (m^2 A + m^2 B + lC) \nabla_r \psi + \psi^\dagger (m^4 B + lm^2 C) \psi\} dV$$

$$= \lambda^2 \int_V \psi^\dagger N\phi \, dV - \frac{m^2 l^*}{|l|^2} \int_V \psi^\dagger F(x) \psi \, dV \qquad \text{...(11)}$$

where $\nabla_r \psi$ represents the matrix $\left[\frac{\partial \psi_i}{\partial x_r}\right]_{n \times 1}$, $i = 1, 2, \ldots, n$ and l^* is the complex conjugate of l.

To evaluate the integral $\int_V \psi^\dagger N\phi \, dV$ occurring in eqn. (11), we take the conjugate transpose of both sides of eqn. (2), right multiply the resulting equation by Φ and integrate over the domain V. This gives

$$\lambda^2 \int_V \psi^\dagger N\phi \, dV = - \lambda^2 \int_V [\nabla^2 \phi^\dagger E^\dagger - \phi^\dagger G^\dagger m^2 - \phi^\dagger H^\dagger_{(x)} l^*] \phi \, dV$$

$$= - \lambda^2 \int_V [\nabla^2 \phi^\dagger E - \phi^\dagger G m^2 - \phi^\dagger H(x) l^*] \phi \, dV$$

$$= + \lambda^2 \int_V [\nabla_r \phi^\dagger E \nabla_r \phi + \phi^\dagger (G m^2 + H(x) l^*) \phi] \, dV \qquad \text{...(12)}$$

where we have made use of Gauss' theorem in n-dimensions and the boundary conditions on ϕ_i.

Substituting for $\lambda^2 \int_V \psi^\dagger N\phi \, dV$ from eqn. (12) in eqn. (11)

we get

$$\int_V [\nabla^2 \psi^\dagger A \nabla^2 \psi + \nabla_r \psi^\dagger (m^2 A + m^2 B + lC) \nabla_r \psi + \psi^\dagger (m^4 B + lm^2 C) \psi] \, dV$$

$$= \lambda^2 \int_V [\nabla_r \phi^\dagger E \nabla_r \phi + \phi^\dagger (G m^2 + H(x) l^*) \phi] \, dV - \frac{m^2 l^*}{|l|^2} \int_V \psi^\dagger F(x) \psi \, dV. \qquad \text{...(13)}$$

Equating the imaginary parts of both sides of eqn. (13), we obtain

$$l_i \int_V [\nabla_r\psi^\dagger C \nabla_r\psi + m^2\psi^\dagger C\psi]\, dV$$

$$= -\lambda^2 l_i \int_V \phi^\dagger H(x)\phi\, dV + \frac{m^2 l_i}{|l|^2}\int_V \psi^\dagger F(x)\,\psi\, dV. \quad \text{...(14)}$$

But since $l_i \neq 0$, eqn. (14) gives

$$\int_V [\nabla_r\psi^\dagger C\nabla_r\psi]\, dV + m^2 \int_V \psi^\dagger\left[C - \frac{F(x)}{|l|^2}\right]\psi\, dV + \lambda^2\int_V \phi^\dagger H(x)\phi\, dV = 0.$$

...(15)

Equation (15) implies that

$$\int_V \psi^\dagger\left[C - \frac{F(x)}{|l|^2}\right]\psi\, dV < 0,$$

which further implies that

$$C_1 \int_V \psi^\dagger\,\psi\, dV - \frac{f_1}{|l|^2}\int_V \psi^\dagger\,\psi\, dV < 0$$

and, therefore, $|l|^2 < \dfrac{f_1}{C_1}$.

This proves the theorem.

We now prove another theorem in which the matrix $F(x)$ of eqn. (1) is not Hermitian, but of the form $F(x) = -il\,J(x)$, where $J(x)$ is an $n \times n$ Hermitian positive definite (everywhere in V) matrix with entries as continuous complex valued function at each point x in V.

Theorem 2 — With the above character of $F(x)$ and other conditions of Theorem 1 remaining the same,

$$0 < l_i < \frac{j_1}{C_1},$$

where j_1 is the largest eigenvalue of $J(x)$ in V.

PROOF : Consider eqns. (1) and (2), (3), where $F(x)$ is of the form stated above. Then proceeding exactly as in the proof of Theorem 1, we have corresponding to eqn. (13), the following equation

$$\int_V [\nabla^2\psi^\dagger A\nabla^2\psi + \nabla_r\psi^\dagger (m^2A + m^2B + lC]\,\nabla_r\psi + \psi^\dagger (m^4B + lm^2C)\,\psi]\, dV$$

$$= \lambda^2 \int_V [\nabla_r\phi^\dagger E\nabla_r\phi + \phi^\dagger (Gm^2 + H(x)\, l^*)\,\phi]\, dV + im^2\int_V \psi^\dagger J(x)\psi\, dV.$$

...(16)

Equating the imaginary parts of both sides of eqn. (16), we obtain

$$l_i \int_V [\nabla_r \psi^\dagger C \nabla_r \psi + m^2 \psi\, C\psi]\, dV = -\lambda^2 l_i \int_V \phi^\dagger H(x) \phi\, dV$$

$$+ m^2 \int_V \psi^\dagger J(x)\, \psi\, dV. \qquad \ldots(17)$$

But since $l_i \neq 0$, eqn. (17) gives

$$\int_V [\nabla_r \psi^\dagger C \nabla_r \psi]\, dV + m^2 \int_V \psi^\dagger \left[C - \frac{J(x)}{l_i}\right] \psi\, dV + \lambda^2 \int_V \phi^\dagger H(x)\, \phi\, dV = 0. \qquad \ldots(18)$$

Equation (18) implies that

$$\int_V \psi^\dagger \left[C - \frac{J(x)}{l_i}\right] \psi\, dV < 0. \qquad \ldots(18')$$

It thus follows that $l_i > 0$. This leads to

$$C_1 \int_V \psi^\dagger \psi\, dV - \frac{j_1}{l_i} \int_V \psi^\dagger \psi\, dV < 0, \text{ and therefore, } 0 < l_i < \frac{j_1}{C_1}.$$

This proves the theorem.

Theorem 3 — If (l, ψ, ϕ), with $l_i \neq 0$, is a solution of equations

$$l\,[\nabla^2 - m^2]\,[A\,\nabla^2 - m^2 B - lC]\,\psi = \lambda^2 l N \phi - m^2 F(x)\,\psi + i \langle | \rangle\, \psi \qquad \ldots(19)$$

and

$$[E\,\nabla^2 - i\,\boxed{\diagdown} - m^2 G - l H(x)]\,\phi = -N\psi \qquad \ldots(20)$$

together with boundary conditions

$$\left.\begin{aligned} &\psi_i = 0 = \phi_i \\ \text{and} \quad &\frac{\partial \psi_i}{\partial n} = 0 \end{aligned}\right\} \qquad \ldots(21)$$

on S, then

$$|\,l\,|^2 < \frac{f_1}{C_1},$$

where the symbols $\langle | \rangle$ and $\boxed{\diagdown}$ stand respectively for the operators

$$\sum_{i=1}^{n} \left[\delta_i \frac{\partial}{\partial x_i} + \gamma_i \frac{\partial^3}{\partial x_i^3}\right] \text{ and } \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial x_i}$$

with δ_i, γ_i and η_i being real constants.

PROOF : We follow exactly the proof of theorem 1 and not that the only additional terms that we now have to consider on the right hand sides of eqns. (11) and (12) are $i \int_V \psi^\dagger \langle | \rangle \psi \, dV$ and $i\lambda^2 \int_V (\underline{|\diagdown|}\, \phi)^\dagger \phi \, dV$ respectively.

$$\int_V \psi^\dagger \langle | \rangle \psi \, dV = \int_V \psi_r^* \langle | \rangle \psi_r \, dV$$

$$= \sum_{i=1}^{n} \int_V \psi_r^* \left[\delta_i \frac{\partial}{\partial x_i} + \gamma_i \frac{\partial^3}{\partial x_i^3} \right] \psi_r \, dV$$

$$= \sum_{i=1}^{n} \int_V \psi_r^* \, \delta_i \frac{\partial \psi_r}{\partial x_i} \, dV + \sum_{i=1}^{n} \int_V \psi_r^* \, \gamma_i \frac{\partial^3 \psi_r}{\partial x_i^3} \, dV.$$

Letting $\psi_r = u_r + iv_r$, where u_r and v_r are real, we have from the above equation

$$\int_V \psi^\dagger \langle | \rangle \psi \, dV = \sum_{i=1}^{n} \delta_i \int_V (u_r - iv_r) \frac{\partial}{\partial x_i} (u_r + iv_r) \, dV$$

$$+ \sum_{i=1}^{n} \int_V (u_r - iv_r) \, \gamma_i \frac{\partial^3}{\partial x_i^3} (u_r + iv_r) \, dV$$

$$= \sum_{i=1}^{n} \delta_i \int_V \tfrac{1}{2} \frac{\partial}{\partial x_i} (u_r^2 + v_r^2) \, dV + i \sum_{i=1}^{n} \delta_i \int_V \left[u_r \frac{\partial v_r}{\partial x_i} - v_r \frac{\partial u_r}{\partial x_i} \right] dV$$

$$+ \sum_{i=1}^{n} \gamma_i \int_V \left[u_r \frac{\partial^3 u_r}{\partial x_i^3} + v_r \frac{\partial^3 v_r}{\partial x_i^3} \right] dV + i \sum_{i=1}^{n} \gamma_i \int_V \left[u_r \frac{\partial^3 v_r}{\partial x_i^3} - v_r \frac{\partial^3 u_r}{\partial x_i^3} \right] dV$$

$$= i \sum_{i=1}^{n} \delta_i \int_V \left[u_r \frac{\partial v_r}{\partial x_i} - v_r \frac{\partial u_r}{\partial x_i} \right] dV + i \sum_{i=1}^{n} \gamma_i \int_V \left[u_r \frac{\partial^3 v_r}{\partial x_i^3} - v_r \frac{\partial^3 u_r}{\partial x_i^3} \right] dV \qquad \ldots(22)$$

the real terms on the right hand side of eqn. (22) vanish on account of the boundary conditions satisfied by ψ_r.

$$\int_V (\underline{|\diagdown|}\, \phi)^\dagger \phi \, dV = \int_V (\underline{|\diagdown|}\, \phi_r^*) \, \phi_r \, dV = \sum_{i=1}^{n} \int_V \eta_i \frac{\partial \phi_r^*}{\partial x_i} \, \phi_r \, dV$$

Letting $\phi_r = u'_r + iv'_r$, where u'_r and v'_r are real, we have from the above equation

$$\int_V (\boxed{\diagdown}\,\phi_r^*)\,\phi_r\,dV = \sum_{i=1}^{n} \eta_i \int_V (u'_r + iv'_r)\frac{\partial}{\partial x_i}(u'_r - iv'_r)\,dV$$

$$= \sum_{i=1}^{n} \eta_i \int_V \tfrac{1}{2}\frac{\partial}{\partial x_i}(u'^2_r + v'^2_r)\,dV + i\sum_{i=1}^{n} \eta_i \int_V \left[v'_r \frac{\partial u'_r}{\partial x_i} - u'_r \frac{\partial v'_r}{\partial x_i}\right] dV$$

$$= i\sum_{i=1}^{n} \eta_i \int_V \left[v'_r \frac{\partial u'_r}{\partial x_i} - u'_r \frac{\partial v'_r}{\partial x_i}\right] dV \qquad \ldots(22')$$

the real term on the right hand side of eqn. (22′) vanishes on account of the boundary conditions satisfied by ϕ_r.

The above calculations show that eqn. (14) remains unaltered.

This proves the theorem.

Remark : We have a similar result corresponding to the $F(x)$ of Theorem 2.

Lemma 1 — If (l, ψ, ϕ) is a solution of eqns. (1), (2) and (3) with E and G positive definite everywhere in V and the set of zeros of F being non-dense in V, then l cannot be zero.

PROOF : If possible, let $l = 0$. Equation (1) then gives $m^2F(x)\,\psi = 0$ everywhere in V, and, therefore, $\psi \equiv 0$ everywhere in V. From eqn. (2), we then have

$$[E\,\nabla^2 - m^2G]\,\phi = 0.$$

The above equation together with the boundary conditions (3) implies that

$$\int_V [\nabla_r\phi^\dagger\, E\,\nabla_r\phi + \phi^\dagger\, Gm^2\phi]\,dV = 0. \qquad \ldots(23)$$

Equation (23) clearly shows that $\phi \equiv 0$ everywhere in V. This proves the lemma.

Remark 2 : In one dimension, a simply connected region is an interval and, therefore, its boundary consists of the two end points of the interval. Further, the various steps which make use of Gauss' theorem in n-dimension in the proof of Theorems 1 and 2 reduce to the usual integration by parts. The operator ∇^{2k} $(k = 1, 2)$ becomes respectively the ordinary second and the fourth derivation w.r.t. the single independent variable, $\partial/\partial n$ at any point on the boundary reduces to the ordinary first derivation with respect to the single independent variable at the two end points of the interval, while the various Hermitian matrices involved in the governing equations become real matrices of order 1×1. Clearly, Theorems 1 and 2 remain true in one dimension, provided we replace the above real matrices of order 1×1 by the corresponding elements of which they are constituted.

Applications to Hydrodynamic Stability

(a) *Stability of Spiral Flows*

The governing differential equations and the boundary conditions of the linear axisymmetric stability problem of a narrow gap viscous Couette flow with an axial pressure gradient at the marginal state are given by (Chandrasekhar 1961).

$$[\{(D^2 - a^2) - i(\sigma + \text{R}a)\}(D^2 - a^2) - 12i\text{R}a]\, u = v \qquad \ldots(24)$$

and $$[D^2 - a^2 - i(\sigma + \text{R}a)]\, v = -\bar{T}a^2 u, \quad -\tfrac{1}{2} < \zeta < +\tfrac{1}{2} \qquad \ldots(25)$$

and $$u = Du = v = 0 \quad \text{for } \zeta = \pm\tfrac{1}{2} \qquad \ldots(26)$$

σ and $\text{R}a$ are real constants and the various other symbols occurring in the above equations have the same meanings as given by Chandrasekhar (1961). We note that $\sigma + \text{R}a$ for the above problem cannot be zero, since it leads to a trivial solution for u and v, as can be easily seen.

We prove the following theorem.

Theorem 4 — If (σ, u, v) with σ = a real constant is a solution of eqns. (24), (25) and (26), then

$$-\text{R}a < \sigma < \frac{12\text{R}}{a} - \text{R}a. \qquad \ldots(27)$$

Proof : Since $\sigma + \text{R}a \neq 0$, the governing differential equations (1) and (2) and boundary conditions (3) of the present paper in one dimension respectively coincide $\left(\text{taking only } \frac{\partial \psi_i}{\partial n} = 0 \text{ on } S\right)$ with eqns. (24), (25) and (26), when the symbols m^2, l, ϕ, ψ and λ^2 are respectively renamed as a^2, $i(\sigma + \text{R}a)$, v, $\bar{T}a^2u$ and $\bar{T}a^2$ with

$$A = B = C = N = E = H(x) = G = [1]_{1\times1},\ F(x) = \left[\frac{12\text{R}}{a}(\sigma + \text{R}a)\right]_{1\times1}$$

and Remark 2 is taken into account.

Clearly, since $F(x) \neq 0$, we must have according to Lemma 1

$$l = i(\sigma + \text{R}a) \neq 0 \text{ and,}$$

therefore, theorem 1 is applicable.

It follows that

$$|\, l \,|^2 < \frac{12\text{R}}{a}(\sigma + \text{R}a),$$

or $$(\sigma + \text{R}a)^2 < \frac{12\text{R}}{a} (\sigma + \text{R}a)$$

for which a necessary condition is $-\text{R}a < \sigma < \frac{12\text{R}}{a} - \text{R}a$. This proves the theorem.

Contents of inequality (27) are completely in accord with the numerical calculations of Chandrasekhar (1960) on the assumption of an overstable marginal state, as mentioned earlier, and summed up in TABLEXL on page 376 of his treatise on Hydrodynamic and Hydromagnetic stability (1961). The result further shows that for $a > \sqrt{12}$, σ is necessarily negative and, therefore, the principle of exchange of stabilities is not satisfied for the problem.

For the present problem, however, it is possible to establish a more general result than contained in Theorem 4 by applying Theorem 2. We then have :

Theorem 5 — If (σ, u, v) with $\sigma = a$ complex constant, in general, is a solution of eqns. (24), (25) and (26), then $-\text{R}a < \sigma_r < \frac{12\text{R}}{a} - \text{R}a$, where σ_r is the real part of σ.

PROOF : The proof goes exactly along the same lines as in the case of Theorem 4, except that now we take

$$F(x) = -\, ilJ(x) = \left[(\sigma + \text{R}a)\left(\frac{12\text{R}}{a}\right)\right]_{1\times 1}, \text{ and } J(x) = \left[\frac{12\text{R}}{a}\right]_{1\times 1}.$$

Theorem 5 thus shows that the contents of Theorem 4 are valid even when σ is a complex constant, i.e., for the non-marginal modes of the system also. Theorem 5 further shows that an arbitrary mode with even number exceeding $\sqrt{12}$ cannot be non-oscillatory and, therefore, in particular, the principle of exchange of stabilities is not valid for the problem.

Such an explicit upper and a lower bound for the frequency of oscillations of an arbitrary mode of the problem and the results derived therein are believed to be given here for the first time.

(b) *Stability of Thermohaline Convection*

The governing differential equations and the boundary conditions of the linear stability problem of thermohaline convection for free as well as rigid boundaries under the assumption that the diffusion of salt is much slower than the diffusion of heat are given by

$$p\left[D^2 - \pi^2\alpha^2 - \frac{p}{\sigma}\right][D^2 - \pi^2\alpha^2]\,\psi = -\, R\pi\alpha p T - R_s\pi^2\alpha^2\psi \qquad \text{...(28)}$$

and $$[D^2 - \pi^2\alpha^2 - p]\, T = \pi\alpha\psi, \quad \text{for } 0 < z < 1 \qquad \text{...(29)}$$

with

$$\left.\begin{array}{lll} \psi = 0 = T \text{ at } z = 0 \text{ and } z = 1 & & \\ \text{and either } D\psi = 0 \text{ at } z = 0 \text{ and } z = 1 & \text{(rigid boundary)} & \\ \text{or } D^2\psi = 0 \text{ at } z = 0 \text{ and } z = 1 & \text{(free boundary)} & \end{array}\right\} \quad \ldots(30)$$

where the operator D stands for d/dz and the various other symbols occurring in the above equations have the same meaning as in the paper of Veronis (1965). It is to be further noted here that we cannot take the z-dependence of the eigen-solutions ψ and T as $\sin n\pi z$, as taken by Veronis, because we are including in our analysis the case of rigid boundaries also.

We prove the following theorem.

Theorem 6 — If (p, ψ, T) with $p = p_r + ip_i$, p_r and p_i being real and $p_i \neq 0$ is a solution of eqns. (28), (29) and (30), then

$$|p|^2 < \sigma R_s.$$

PROOF : We note that the governing eqns. (1) and (2) and the boundary conditions (3) of the present paper in one dimension respectively coincide with eqns. (28), (29) and (30), when the symbols m^2, l, ϕ, ψ and λ^2 are respectively renamed as

$$\pi^2\alpha^2, \frac{p}{\sigma}, \ -T, \ \pi\alpha\psi \text{ and } R\pi^2\alpha^2$$

with $$A = B = C = N = E = G = [1]_{1\times 1},$$

$$H(x) = [\sigma]_{1\times 1} \text{ and } F(x) = \left[\frac{R_s}{\sigma}\right]_{1\times 1},$$

and Remark 2 is taken into account.

Clearly, since $p_i \neq 0$, Theorem 1 is applicable.

It follows that

$$\frac{|p|^2}{\sigma^2} < \frac{R_s}{\sigma}$$

or $$|p|^2 < \sigma R_s,$$

and hence the theorem is proved.

If we consider the equations of Veronis at the marginal state, then in this notation $p = ip_m$ with p_m real. Theorem 6 is still valid, since $p_m \neq 0$ (by Lemma 1) and we get

$$p_m^2 < \sigma R_s. \quad \ldots(31)$$

The contents of inequality (31) are completely in accord with the calculations of Veronis, though his solution is surely restricted by the fact that he has analysed the problem for the case of free boundaries only. The calculations of p_m's in the case of rigid boundaries for the present problem get extremely complicated and in such situations the result is a new contribution to the field.

(c) *Stability of Thermal Convection*

The governing differential equations and the boundary conditions of the linear stability problem of thermal convection for free as well as rigid boundaries are given by Chandrasekhar (1961)

$$[D^2 - a^2][D^2 - a^2 - \sigma] W = \left(\frac{g\alpha d^2}{\nu}\right) a^2 \circledR \qquad \ldots(32)$$

and $$[D^2 - a^2 - p\sigma] \circledR = -\left(\frac{\beta d^2}{\kappa}\right) W, \text{ for } 0 < z < 1 \qquad \ldots(33)$$

with $W = 0 = \circledR$ at $z = 0$ and $z = 1$

and either $DW = 0$ at $z = 0$ and $z = 1$ (rigid boundary)

or $D^2 W = 0$ at $z = 0$ and $z = 1$ (free boundary). ...(34)

The various symbols occurring in the above equations have the same meanings as given by Chandrasekhar (1961). We prove the following theorem.

Theorem 7 — If $(\sigma, W, \circledR)$ with $\sigma = \sigma_r + i\sigma_i$, σ_r and σ_i being real constants is a solution of eqns. (32), (33) and (34), then

$$\sigma_i = 0.$$

PROOF: Let if possible $\sigma_i \neq 0$. Then the governing eqns. (1) and (2) and the boundary conditions (3) of the present paper in one dimension respectively coincide with eqns. (32), (33) and (34), when the symbols m^2, l, ϕ, ψ and λ^2 are respectively renamed as a^2, σ, $\circledR$, W and $\left(\frac{g\alpha a^2\kappa}{\nu\beta}\right)$ with

$$A = B = C = E = G = [1]_{1\times1}; \; N = \left[\frac{\beta d^2}{\kappa}\right]_{1\times1}, \; H(x) = [p]_{1\times1} \text{ and}$$

$$F(x) = [0]_{1\times1},$$

and Remark 2 is taken into account.

But since $\sigma_i = l_i \neq 0$, we must have according to Theorem 1,

$$| l_i |^2 = | \sigma_i |^2 < 0$$

which is a contradiction. Hence, the theorem is proved. The above result is the famous result due to Pellew and Southwell (1940).

(d) *Stability of Rayleigh-Taylor Configuration*

The governing differential equations and the boundary conditions of the linear stability problem of a Rayleigh-Taylor configuration of a Boussinesq fluid of constant coefficient of viscosity are given by Banerjee and Kalthia (1971).

$$n\,[D^2 - k^2]\left[D^2 - k^2 - \frac{\rho_0 n}{\mu}\right] w = \frac{gk^2}{\mu}\,(D\rho)\,w, \quad 0 < z < d \qquad \text{...(35)}$$

with $\quad w = 0 = Dw$ at $z = 0$ and $z = d \qquad$ ...(36)

where the various symbols occurring in the above equations have the same meanings as given by Banerjee and Kalthia (1971). We prove the following theorem.

Theorem 8 — If (n, w) with $n = n_r + in_i$, n_r and n_i being real constants and $n_i \neq 0$, is a solution of eqns. (35) and (36), then

$$|\,n\,|^2 < \max_{\text{Flow Domain}} \left\{\frac{-gD\rho}{\rho_0}\right\}. \qquad \text{...(37)}$$

PROOF: To establish the above, we first note that a necessary condition for the existence of $n_i \neq 0$ is that $D\rho$ must be negative at least at one point in the open interval $(0, d)$. Further, Theorem 2 remaining valid even if $\lambda^2 = 0$ and eqn. (2) together with boundary condition on ϕ_i are ignored, as can be easily seen from the proof of the theorem. Then eqn. (1) and boundary conditions (3) of the present paper in one dimension respectively coincide with eqns. (35) and (36), when the symbols m^2, l, ψ are respectively renamed as k^2, n, w with

$$A = B = [1]_{1\times1}; \quad C = \left[\frac{\rho_0}{\mu}\right]_{1\times1}; \quad F(x) = \left[\frac{-gD\rho}{\mu}\right]_{1\times1}$$

and Remark 2 is taken into account.

But since $n_i \neq 0$, Theorem 1 is applicable and it follows that

$$|\,n\,|^2 < \max_{\text{Flow Domain}} \left\{\frac{-gD\rho}{\rho_0}\right\}$$

and hence the theorem is proved.

The contents of inequality (37) are completely in accord with the theorem proved by Banerjee and Kalthia (1971), though their theorem is more general in the sense that they establish the result even for a non-Boussinesq fluid of variable coefficient of viscosity.

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