UNSTEADY FLOW OF A MAXWELL FLUID PAST A FLAT PLATE

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This paper presents a theoretical study of the flow pattern set up in a linear visco-elastic fluid past an infinite flat plate when the plate is moving parallel to itself with an arbitrary time dependent velocity. The pressure is assumed to be uniform with the initial velocity distribution in an exponential form. Using transform techniques, exact solutions are obtained for the velocity profiles of the flow field in the following cases : (a) When the plate is moving in its own plane harmonically with time and (b) when the velocity of the plate is decaying exponentially with time. The effect of the relaxation parameter of the fluid on its behaviour has been studied and the dependence of the velocity on the initial distribution of velocity and on the motion of the plate has been determined. Several particular cases can be easily deduced and discussed on the basis of the results obtained.

1. INTRODUCTION

The study of the motion engendered in a fluid by the action of a solid body moving relative to it is of interest not only for its own sake, but also because it illustrates the effect of specific boundary and initial conditions on the properties of the motion. Although the first theoretical attempt to solve a problem of this type was made by Stokes (1850), who studied the effect of viscosity in modifying the motion of a fluid in contact with vibrating solids, such problems are still attracting the attention of workers in the field. Na and Sidhom (1967) obtained some interesting results for the flow of visco-elastic fluids of the Maxwell type near an accelerating or oscillating plate.

Prakash (1971) considered the corresponding problem for ordinary viscous fluids under generalized boundary conditions with the initial distribution of velocity in an exponential form.

In the present paper, the above problem is discussed for elastico-viscous fluids of the Maxwell type. It is assumed that the fluid properties are uniform in time and space, the flow is laminar, the surface is smooth and the "no slip" condition exists at the surface, i.e., the fluid in the immediate vicinity of the surface moves with the surface. The plate is moving parallel to itself with an arbitrary time dependent velocity under uniform pressure and the velocity distribution in the initial state is assumed to be in an exponential form. Exact theoretical solutions for velocity profiles have been derived in a general form by the joint Laplace and Fourier sine transform treatment. A few particular solutions of the problem related to certain velocity distributions of physical interest are also presented.

It has been found that, unlike an ordinary viscous fluid, the disturbance due to the plate motion is not instantaneous in the whole of the space, but it has a finite velocity of propagation. In addition, it is observed that the relaxation parameter increases the magnitude of the velocity, and produces distortion in the propagation of the disturbance. The solution of the corresponding problem for ordinary viscous fluids has been recovered as a special case. The results are shown to be in complete agreement with those given by Prakash (1971).

2. Required Integral Transforms

The Laplace transform of any arbitrary function $f(t)$ is defined as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \ \mathrm{Re}\ s > 0 \qquad \ldots(2.1)$$

provided the above integral exists.

Also, if the Fourier sine transform of $\varphi(x, t)$ be

$$F(\xi, t) = \int_0^\infty \varphi(x, t) \sin \xi x\, dx, \ \xi > 0 \qquad \ldots(2.2)$$

then we have (Churchill 1972)

$$\varphi(x, t) = \frac{2}{\pi} \int_0^\infty F(\xi, t) \sin \xi x\, dx, \ x > 0. \qquad \ldots(2.3)$$

3. Basic Equation and Statement of the Problem

For a linear, isotropic, visco-elastic fluid, the stress tensor is given by (Fredrickson 1964)

$$S^{ij} = -pg^{ij} + p^{ij} \qquad \ldots(3.1)$$

where p is the static pressure; g^{ij}, the associated metric tensor and p^{ij}, a tensor usually related to the rate of strain, e^{ij}, by the "equation of state"

$$P' p^{ij} = 2Q' e^{ij} \qquad \ldots(3.2)$$

P' and Q' are two operators defined by

$$P' = 1 + \lambda_0 \frac{d}{dt} + \lambda_1 \frac{d^2}{dt^2} + \dots + \lambda_n \frac{d^{n+1}}{dt^{n+1}} \quad \dots(3.3)$$

and

$$Q' = \mu \left(1 + s_0 \frac{d}{dt} + s_1 \frac{d^2}{dt^2} + \dots + s_n \frac{d^{n+1}}{dt^{n+1}}\right). \quad \dots(3.4)$$

The quantity μ in eqn. (3.4) is the viscosity of the material at zero rate of shear, $\lambda_0, \lambda_1, \dots, \lambda_n; s_0, s_1, \dots, s_n$ are considered as physical constants, and d/dt is the convected derivative.

By definition, the type of fluids known as Maxwell fluids obey the following equation of state :

$$\left(1 + \lambda_0 \frac{d}{dt}\right) p^{ij} = 2\mu e^{ij}. \quad \dots(3.5)$$

Equation (3.5) is a special case of eqn. (3.2). The physical significance of λ_0 (the relaxation time) is that if the motion stops suddenly, the shear stress will decay as $\exp\left(-\frac{t}{\lambda_0}\right)$.

For unsteady parallel flow of Maxwell fluids over a flat plate, infinite in extent, at the $y = 0$ plane, we consider the motion of the plate to be in its own plane, so that there is no displacement of the fluid in the y-direction due to the plate motion and, therefore, the velocity of the fluid in the y-direction is zero. Further, since the plate extends to infinity in the $\pm$ x-directions, there are no changes in the x-direction of any property.

The momentum equation representing the motion of a Maxwell fluid can be written as

$$\rho\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial y}(p_{xy}) \quad \dots(3.6)$$

where u is the component of the fluid velocity in the direction of the x-axis, ρ the density of the fluid and p_{xy} the physical component of p^{ij}, defined by

$$\left(1 + \lambda_0 \frac{\partial}{\partial t}\right) p_{xy} = \mu \frac{\partial u}{\partial y}. \quad \dots(3.7)$$

The initial and boundary conditions for the problem under cosideration are :

$$\text{(i)} \quad u = Ae^{-By} \text{ at } t = 0 \text{ for } y \geqslant 0 \quad \dots(3.8)$$

$$\left.\begin{aligned} &\text{(ii)} \quad u = g(t) \quad \text{at } y = 0 \text{ for } t > 0 \\ &\text{(iii)} \quad \lim_{y \to \infty} [u(y, t)] = 0 \text{ for } t \geqslant 0. \end{aligned}\right\} \quad \dots(3.9)$$

Here, A and B are non-negative constants and $g(t)$ is a bounded continuous or piece-wise continuous arbitrary function of t.

4. Solution of the Problem

On taking Laplace transform defined by (2.1), eqns. (3.6) and (3.7), subject to the initial condition (3.8), transform to

$$\rho(s\bar{u} - Ae^{-By}) = \frac{d}{dy}\bar{p}_{xy} \qquad \ldots(4.1)$$

and

$$(1 + \lambda_0 s)\,\bar{p}_{xy} = \mu \frac{d\bar{u}}{dy} \qquad \ldots(4.2)$$

where

$$\bar{p}_{xy} = \int_0^\infty e^{-st}\, p_{xy}\, dt$$

and

$$\bar{u} = \int_0^\infty e^{-st}\, u(y, t)\, dt.$$

It has been assumed that p_{xy} vanishes initially. Now eliminating p_{xy} from (4.1) and (4.2), we obtain

$$\nu \frac{d^2\bar{u}}{dy^2} - (\lambda_0 s^2 + s)\,\bar{u} = -\,Ae^{-By}\,(1 + \lambda_0 s) \qquad \ldots(4.3)$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematical coefficient of viscosity.

The boundary conditions are transformed into

$$\bar{u} = \bar{g}(s) \text{ at } y = 0$$

$$\bar{u} \to 0 \text{ as } y \to \infty.$$

Solving (4.3), say, by Fourier sine transform and letting

$$p^2 = \frac{s(1 + \lambda_0 s)}{\nu} \qquad \ldots(4.4)$$

we get

$$\bar{u} = \left[\bar{g}(s) - \frac{A(1 + \lambda_0 s)}{\nu(p^2 - B^2)}\right] e^{-py} + \frac{(1 + \lambda_0 s)\,Ae^{-By}}{\nu(p^2 - B^2)},\ y > 0. \qquad \ldots(4.5)$$

Finally, Laplace inversion of (4.5) leads to (Carslaw and Jaeger 1953)

$$u(y,t) = \frac{A}{\lambda_0(\mu_1 - \mu_2)} \left[(1 + \lambda_0\mu_1)\, e^{-By+\mu_1 t} - (1 + \lambda_0\mu_2)\, e^{-By+\mu_2 t}\right]$$

$$\text{for } 0 \leqslant t \leqslant \frac{y}{V}$$

and

$$= g\left(t - \frac{y}{V}\right) e^{-\sigma\cdot(y/V)} + \frac{\sigma y}{V}\int_{y/V}^{t} g(t-\tau)\, e^{-\sigma\tau}\cdot \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}}\, d\tau$$

$$+ \frac{A}{\lambda_0(\mu_1 - \mu_2)}\Bigg[-(1 + \lambda_0\mu_1)\Bigg\{e^{\mu_1 t - (\mu_1+\sigma)y/V}$$

$$+ \frac{\sigma y}{V}\int_{y/V}^{t} e^{\mu_1 t - (\mu_1+\sigma)\tau}\cdot \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}}\, d\tau\Bigg\}$$

$$+ (1 + \lambda_0\mu_2)\Bigg\{e^{\mu_2 t - (\mu_2+\sigma)y/V}$$

$$+ \frac{\sigma y}{V}\int_{y/V}^{t} e^{\mu_2 t - (\mu_2+\sigma)\tau}\cdot \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}}\, d\tau\Bigg\}$$

$$+ \Bigg\{(1 + \lambda_0\mu_1)\, e^{-By+\mu_1 t} - (1 + \lambda_0\mu^2)\, e^{-By+\mu_2 t}\Bigg\}\Bigg]$$

$$\text{for } t > \frac{y}{V} \qquad \ldots(4.6)$$

where

$$\sigma = \frac{1}{2\lambda_0}$$

$$V = \sqrt{\frac{\nu}{\lambda_0}}$$

I_1 denotes the modified Bessel function of the first kind and of order unity and μ_1 and μ_2 are the roots of the equation

$$\lambda_0 s^2 + s - \nu B^2 = 0$$

given by

$$\mu_1, \mu_2 = \frac{-1 \pm \sqrt{1 + 4\lambda_0 \cdot \nu B^2}}{2\lambda_0}.$$

Now we discuss the character of the motion exhibited by the above solution given by (4.6). We see that the first term of the second row corresponds to a propagation of the input disturbance in the direction of increasing y with velocity V. This implies that the disturbance produced by an oscillating plane in a visco-elastic fluid is not instantaneous, as in the ordinary viscous case, but it has a finite velocity of propagation. The velocity amplitude falls off exponentially with distance from the plate, having dropped by a factor.

$$e^{-1} = 0.3679$$

over a distance

$$y = \frac{V}{\sigma}.$$

The second term in the second row represents the cumulative effect of the driving function $g(\tau)$ during interval $\tau = 0$ to $\tau = \left(t - \frac{y}{V}\right)$, i.e., it gives the sum of all the values from $g(0)$ to $g\left(t - \frac{y}{V}\right)$ of the input function which have arrived at the point y up to the time t. It means that at a point (y, t), all values of the driving force which have arrived previously and remained there as residues are superimposed.

The third term of the same row and the expression in the first row of the above result may be recognized as the contribution of the initial distribution.

In the case of ordinary viscous fluid motion

$$\sigma \rightarrow \infty$$

and the result (4.6), in view of the asymptotic expansion of $I_1(z)$ for real and large z (Bateman 1953), reduces to

$$u(y, t) = \left[\frac{2}{\sqrt{\pi}} \left\{ \int_0^\infty g\left(t - \frac{y^2}{4\nu\eta^2}\right) e^{-\eta^2} \, d\eta - \right.\right.$$

(*equation continued on p.* 163)

$$- \int_0^{y/\sqrt{4\nu t}} g\left(t - \frac{y^2}{4\nu\eta^2}\right) e^{-\eta^2} \, d\eta\Big\}$$

$$+ \frac{2Ae^{\nu B^2 t}}{\sqrt{\pi}} \int_0^{y/\sqrt{4\nu t}} e^{-(B^2 y^2/4\eta^2)-\eta^2} \, d\eta\Big]$$

$$\text{for } y > 0$$

and

$$u = g(t) \qquad \text{for } y = 0. \qquad \ldots(4.7)$$

The above result (4.7) is in complete agreement with that obtained by Prakash (1971).

It may be seen that the magnitude of the velocity increases due to the non-Newtonian effect. There is some sort of discontinuity in the flow at $y = 0$ at the time of start of the motion, unless $A = g(0)$, as is clear from the initial and the boundary conditions.

It is interesting to note that this discontinuity smoothens out in the non-Newtonian case and the solution (4.6) holds for $y = 0$ also. It is also clear that the initial distribution modifies the velocity profiles considerably in this case. Thus, the velocity is dependent on both the motion of the plate and the initial distribution of the velocity.

5. Particular Cases

(i) *Motion under Harmonic Input*

Let us assume that

$$g(t) = U_0 \cos \omega t \qquad \ldots(5.1)$$

where U_0 and ω are positive constants

Substituting in (4.6), the velocity profiles at any point y at time t are given by

$$u(y, t) = \frac{A}{\lambda_0(\mu_1 - \mu_2)} \left[(1 + \lambda_0 \mu_1) e^{-By+\mu_1 t} - (1 + \lambda_0 \mu_2) e^{-By+\mu_2 t} \right],$$

$$\text{for } 0 \leqslant t < \frac{y}{V}$$

and

$$u(y, t) = U_0 \cos \omega \left(t - \frac{y}{V}\right) e^{-\sigma y/V} +$$

(*equation continued on p.* 164)

$$+ \frac{\sigma y}{V} \int_{y/V}^{t} U_0 \cos \omega (t - \tau)\, e^{-\sigma \tau} . \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}}\, d\tau$$

$$+ \frac{A}{\lambda_0(\mu_1 - \mu_2)} \left[-(1 + \lambda_0 \mu_1) \left\{ e^{-\mu_1 t - (\mu_1 + \sigma)\, y/V} \right.\right.$$

$$\left. + \frac{\sigma y}{V} \int_{y/V}^{t} e^{\mu_1 t - (\mu_1 + \sigma)\tau} . \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}}\, d\tau \right\}$$

$$+ (1 + \lambda_0 \mu_2) \left\{ e^{\mu_2 t - (\mu_2 + \sigma)\, y/V} \right.$$

$$\left. + \frac{\sigma y}{V} \int_{y/V}^{t} e^{\mu_2 t - (\mu_2 + \sigma)\tau} . \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}}\, d\tau \right\}$$

$$\left. + \left\{ (1 + \lambda_0 \mu_1)\, e^{-By + \mu_1 t} - (1 + \lambda_0 \mu_2)\, e^{-By + \mu_2 t} \right\} \right]$$

$$\text{for } t > \frac{y}{V} \cdot \qquad \ldots(5.2)$$

The first term in the second row shows that in addition to the variation of amplitude with y, there is also a variation in the phase of the velocity wave.

The other terms may be interpreted in a similar way, as in the previous section.

Corresponding results for ordinary viscous fluid motion may be obtained by putting $\lambda_0 = 0$. The result of Prakash (1971) and Batchelor (1967) follows as a very special case of our result by giving suitable values to the parameters involved.

The result for a plate moving suddenly in its own plane with a constant velocity U_0 in the fluid at rest is obtained from (5.2) by taking

$$\omega = 0$$

and

$$A = 0$$

(ii) *Motion under Exponentially Decaying Input*

We now consider the case in which the plate is subjected to a motion decaying exponentially with time.

Let us take

$$g(t) = U_0 e^{-\omega t}. \qquad \ldots(5.3)$$

Then (4.6) becomes

$$u(y,t) = \frac{A}{\lambda_0(\mu_1 - \mu_2)} \left[(1 + \lambda_0 \mu_1) e^{-By + \mu_1 t} - (1 + \lambda_0 \mu_2) e^{-By + \mu_2 t} \right]$$

$$\text{for } 0 \leqslant t \leqslant \frac{y}{V}$$

and

$$= U_0 e^{-\omega(t-(y/V)) - \sigma(y/V)}$$

$$+ \frac{\sigma y}{V} \int_{y/V}^{t} U_0 e^{-\omega(t-\tau) - \sigma\tau} . \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}} d\tau$$

$$+ \frac{A}{\lambda_0(\mu_1 - \mu_2)} \Bigg[- (1 + \lambda_0 \mu_1) \Bigg\{ e^{\mu_1 t - (\mu_1 + \sigma)(y/V)}$$

$$+ \frac{\sigma y}{V} \int_{y/V}^{t} e^{\mu_1 t - (\mu_1 + \sigma)\tau} . \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}} d\tau \Bigg\}$$

$$+ (1 + \lambda_0 \mu_2) \Bigg\{ e^{\mu_2 t - (\mu_2 + \sigma)(y/V)}$$

$$+ \frac{\sigma y}{V} \int_{y/V}^{t} e^{\mu_2 t - (\mu_2 + \sigma)\tau} . \frac{I_1\left[\sigma\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}\right]}{\left(\tau^2 - \frac{y^2}{V^2}\right)^{1/2}} d\tau \Bigg\}$$

$$+ \Bigg\{ (1 + \lambda_0 \mu_1) e^{-By + \mu_1 t} - (1 + \lambda_0 \mu_2) e^{-By + \mu_2 t} \Bigg\} \Bigg]$$

$$\text{for } t > \frac{y}{V}. \qquad \ldots(5.4)$$

Results for the ordinary viscous case may be obtained by specializing the parameters appropriately.

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