ON THE UNIFORM NÖRLUND SUMMABILITY OF LEGENDRE SERIES

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The (C, α) and (N, q_n) summabilities of Legendre series have been discussed by a number of workers, viz. Hobson (1909), Haar (1911), Chapman (1912), Plancherel (1914), Kogebetliatz (1942) and Tripathi (1977), but the study of Legendre series by the uniform Nörlund summability method does not seem to have been attempted. In an attempt to make an advance in this direction, a theorem on the uniform Nörlund summability of Legendre series, has been established in this paper.

§1. The Legendre series, associated with Lebesgue integrable function in the interval defined by $-1 \leqslant x \leqslant 1$, is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \qquad \ldots(1.1)$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{+1} f(x) \, P_n(x) \, dx$$

and the nth Legendre polynomial $P_n(x)$ is defined by the following expansion:

$$\frac{1}{\sqrt{(1 - 2xz + z^2)}} = \sum_{n=0}^{\infty} P_n(x) \, Z^n.$$

We use the following notation :

$$\psi(t) = \psi(\theta, t) = f\{\cos(\theta - t)\} - f(\cos\theta)$$

$$\Psi(t) = \int_0^t | \psi(u) | \, du.$$

§2. Let $\{q_n\}$ be a sequence of constants real or complex, such that

$$Q_n = q_0 + q_1 + \ldots + q_n \neq 0.$$

The sequence to sequence transformation

$$t_n(x) = \frac{1}{Q_n} \sum_{\nu=0}^{n} q_{n-\nu} \, U_\nu(x), \quad (Q_n \neq 0) \qquad \ldots(2.1)$$

where $\sum_{\nu=0}^{\infty} u_\nu(x)$ is an infinite series and

$$U(x) = u_0(x) + u_1(x) + \ldots + u_n(x)$$

defines the sequence $\{t_n(x)\}$ of Nörlund means (Nörlund 1919, Woronoi 1932) of the sequence $U_n(x)$, generated by the sequence of coefficients q_n.

If there exists a function $U = U(x)$, such that

$$\lim_{n\to\infty} [t_n(x) - U] = 0 \qquad \ldots(2.2)$$

uniformly in a set E, then we say that the series $\Sigma\, u_n(x)$ is summable uniformly in E to the sum U.

The regularity conditions of (N, q_n) method are (Hardy 1949)

$$\lim_{n\to\infty} \frac{q_n}{Q_n} = 0 \qquad \ldots(2.3)$$

and

$$\sum_{k=0}^{n} | q_k | = O(| Q_n |). \qquad \ldots(2.4)$$

If it is assumed that $\{q_n\}$ is real, non-negative and monotonic increasing sequence, then the transformation defined by (2.1) is regular.

§3. In this paper, we establish the following :

Theorem — Let $\epsilon(x)$ be a non-negative function of x such that $\epsilon(x)/(x \log x)$ is monotonic and $\epsilon(x)/\log x = O(1)$, as $x \to \infty$. If

$$\sum_{\nu=2}^{n} \frac{\epsilon(\nu)\, Q_\nu}{\nu \log \nu} = O(Q_n),$$

as $n \to \infty$ and

$$\int_0^t | f(x \pm u) - f(x) |\, dx = o\left[\frac{t\,\epsilon\left(\frac{1}{t}\right)}{\log \frac{1}{t}}\right] \qquad \ldots(3.1)$$

uniformly in a set E, defined in the interval $(-1, 1)$ in which $f(x)$ is bounded as $t \to +0$, then the series (1.1) is summable (N, q_n) uniformly in E to the sum $f(x)$, where $\{q_n\}$ is a sequence of real, non-negative and non-increasing constants, such that Q_n tends to infinity with n.

§4. The following lemmas are used in proving our theorem.

Lemma 1 (Sansone 1959) —

$$\sum_{\nu=1}^{n} (2\nu + 1) P_\nu(x) P_\nu(y) = (n + 1) \frac{[P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)]}{(y - x)}. \quad \ldots(4.1)$$

This identity is known as Christoffel's formula of summation.

Lemma 2 — Under the condition (3.1), we have

$$\int_0^t | f\{\cos (\theta - \nu)\} - f(\cos \theta) | \, d_\nu = o\left[\frac{t\epsilon\left(\frac{1}{t}\right)}{\log \frac{1}{t}}\right] \quad \ldots(4.2)$$

as $t \to +0$, where $x = \cos \theta$, $x + u = \cos \varphi$ and $\theta - \varphi = \nu$.

This lemma can be proved on the lines of Foá (1943).

Lemma 3 (McFadden 1942) — For $0 \leqslant a < b \leqslant \infty$; $0 < t \leqslant \pi$ and any n.

$$| \sum_{k=a}^{b} q_k e^{i(n-k)t} | \leqslant CQ_\tau \quad \ldots(4.3)$$

where C is an absolute constant and $\tau = \left[\frac{1}{t}\right]$

§5. Proof of the Theorem : The nth partial sum of the series (1.1) is

$$S_n(x) = \sum_{\nu=0}^{n} a_\nu P_\nu(x)$$

$$= \frac{(n + 1)}{2} \int_{-1}^{+1} f(y) \frac{[P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)]}{(y - x)} dy$$

(by Lemma 1).

Putting $f(y) \equiv 1$, it can be easily seen that

$$1 = \frac{(n + 1)}{2} \int_{-1}^{+1} \frac{[P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)]}{(y - x)} dy.$$

Therefore,

$$S_n(x) - f(x) = \tfrac{1}{2}(n+1) \int_{-1}^{+1} [f(y) - f(x)] \times \frac{[P_{n+1}(y)\, P_n(x) - P_n(y)\, P_{n+1}(x)]}{(y-x)}\, dy$$

and so

$$S_{n-k}(x) - f(x)$$

$$= \tfrac{1}{2}(n-k+1) \int_{-1}^{+1} [f(y) - f(x)] \frac{[P_{n-k+1}(y)\, P_{n-k}(x) - P_{n-k}(y)\, P_{n-k+1}(x)]}{(y-x)}\, dy.$$

Let us take a positive number s, less than 1, and consider it as the sum of the two other positive numbers μ and δ. Let d be another positive number, such that $0 < d < \mu$ and μ_x and μ_x' be two continuous functions of x within $(-1, +1)$, which lie within the limits $d \leqslant \mu_x \leqslant \mu$, $d \leqslant \mu_x' \leqslant \mu$.

Therefore, for $-1 + s \leqslant x \leqslant 1 - s$, we have

$$S_{n-k}(x) - f(x) = \tfrac{1}{2}(n-k+1)\left[\int_{-1}^{x-\mu_x} + \int_{x-\mu_x}^{x+\mu_x'} + \int_{x+\mu_x'}^{+1}\right][f(y) - f(x)]$$

$$\times \frac{[P_{n-k+1}(y)\, P_{n-k}(x) - P_{n-k}(y)\, P_{n-k+1}(x)]}{(y-x)}\, dy,$$

$$= A_{n-k}(x) + B_{n-k}(x) + C_{n-k}(x), \text{ say.} \qquad \ldots(5.1)$$

Hobson (1931) has shown that for $-1 + s \leqslant x \leqslant 1 - s$,

$$\lim_{n\to\infty} A_{n-k}(x) = 0 \qquad \ldots(5.2)$$

and

$$\lim_{n\to\infty} C_{n-k}(x) = 0 \qquad \ldots(5.3)$$

uniformly in the set E.

Now we suppose that $x = \cos\theta$, $y = \cos\varphi$, $0 < \theta < \pi$, $0 < \varphi < \pi$, $1 - \delta = \cos\rho$, $\rho = 1 - (\mu + \delta) = 1 - s = \cos(\rho + \sigma)$, $\rho > 0$, $\sigma > 0$, we have

$$0 < \rho < \frac{\pi}{2}, \; \rho + \sigma < \frac{\pi}{2}.$$

Thus, if η denotes the minimum of [arc cos u — arc cos $(u + \mu)$] for u in $(-1, 1-\mu)$, we have on the lines of Sansone (1959)

$$B_{n-k}(\cos\theta) = \tfrac{1}{2}(n-k+1)\int_{\theta-\eta}^{\theta+\eta}[f(\cos\varphi) - f(\cos\theta)]$$

$$\times \frac{[P_{n-k+1}(\cos\varphi)P_{n-k}(\cos\theta) - P_{n-k}(\cos\varphi)P_{n-k+1}(\cos\theta)]}{(\cos\varphi - \cos\theta)}\sin\varphi\, d\varphi$$

in which $\rho + \sigma \leqslant \theta \leqslant \pi - (\rho + \sigma);\ 0 < \eta \leqslant \sigma$.

With successive transformation, we obtain

$$B_{n-k}(\cos\theta) = D_{n-k}(\theta) + E_{n-k}(\theta);\ \text{say,} \qquad \ldots(5.4)$$

where

$$D_{n-k}(\theta) = \frac{1}{2\pi\sqrt{\sin\theta}}\int_{\theta-\eta}^{\theta+\eta}\frac{[f(\cos\varphi) - f(\cos\theta)]}{\sin\frac{1}{2}(\theta-\varphi)}$$

$$\times \sin(n-k+1)(\theta-\varphi)\sqrt{\sin\varphi}\, d\varphi$$

and obviously

$$E_{n-k}(\theta) = o(1) \qquad \ldots(5.5)$$

as $n \to \infty$, uniformly when x lies within $(-1 + s, 1 - s)$, i.e., in the set E.

Putting $\theta - \varphi = t$, we get

$$D_{n-k}(\theta) = \frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\frac{f\{\cos(\theta-t)\} - f(\cos\theta)}{\sin\frac{1}{2}t}$$

$$\times \sin(n-k+1)t\sqrt{\sin(\theta-t)}\, dt. \qquad \ldots(5.6)$$

So, we have from (5.1) to (5.6)

$$S_{n-k}(x) - f(x)$$

$$= \frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta}\frac{f\{\cos(\theta-t)\} - f(\cos\theta)}{\sin\frac{1}{2}t}$$

$$\times \sin(n-k+1)t\sqrt{\sin(\theta-t)}\, dt + o(1)$$

since $f(x)$ is bounded on the set E, $o(1)$ will tend to zero for any x uniformly in E.

Now,

$$\frac{1}{Q_n}\sum_{k=0}^{n} q_k \{S_{n-k}(x) - f(x)\}$$

$$= \frac{1}{Q_n}\sum_{k=0}^{n} q_k \frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin\frac{1}{2}t}$$

$$\times \sin(n-k+1)\,t\sqrt{\sin(\theta - t)}\,dt + o(1), \text{ uniformly in } E.$$

$$= \frac{1}{\pi\sqrt{\sin\theta}}\int_0^{\eta} [f\{\cos(\theta - t)\} - f(\cos\theta)]\sqrt{\sin(\theta - t)}$$

$$\times \frac{1}{Q_n}\sum_{k=0}^{n} q_k \frac{\sin(n-k+1)}{\sin\frac{1}{2}t}\,dt + o(1), \text{ uniformly in } E.$$

$$= O\left[\int_0^{\eta} |\psi(t)|\ |N_n(t)|\,dt\right] + o(1), \text{ uniformly in } E.$$

$$= O\left[\int_0^{n^{-1}} |\psi(t)|\ |N_n(t)|\,dt\right] + O\left[\int_{n^{-1}}^{\eta} |\psi(t)|\ |N_n(t)|\,dt\right] + o(1).$$

$$= I + J + o(1) \text{ say} \qquad \ldots(5.7)$$

where

$$N_n(t) = \frac{1}{Q_n}\sum_{k=0}^{n} q_k \frac{\sin(n-k+1)t}{\sin\frac{1}{2}t}.$$

To prove the theorem, we have to show that, under our assumptions uniformly in E,

$$I + J = o(1), \text{ as } n \to \infty.$$

Now uniformly in $o < t < n^{-1}$

$$N_n(t) = \frac{1}{Q_n}\sum_{k=0}^{n} q_k \frac{2(n-k+1)\,|\sin\frac{1}{2}t|}{|\sin\frac{1}{2}t|}$$

$$= O(n).$$

Hence,

$$I = O\left[n \int_0^{n^{-1}} |\psi(t)|\, dt\right]$$

$$= o\left[n \frac{1}{n} \frac{\epsilon(n)}{\log n}\right]$$

$$= o(1)$$

uniformly in E, by the condition of the theorem and by Lemma 2.

Again, for $n^{-1} \leqslant t \leqslant \eta$

$$N_n(t) = O\left[\frac{1}{Q_n \sin \frac{1}{2} t} \sum_{k=0}^{n} q_k \sin(n-k+1)\, t\right]$$

$$= O\left[\frac{Q_\tau}{t\, Q_n}\right],$$

by virtue of Lemma 3.

Also,

$$J = O\left[\frac{1}{Q_n} \int_{n^{-1}}^{\eta} |\psi(t)| \frac{Q_\tau}{t}\, dt\right].$$

Now,

$$\frac{1}{Q_n} \int_{n^{-1}}^{\eta} |\psi(t)| \frac{Q_\tau}{t} dt = \frac{1}{Q_n} \left[\Psi(t) \frac{Q_\tau}{t}\right]_{n^{-1}}^{\eta}$$

$$+ \frac{1}{Q_n} \int_{n^{-1}}^{\eta} \frac{\Psi(t)\, Q_\tau}{t^2}\, dt - \frac{1}{Q_n} \int_{n^{-1}}^{\eta} \Psi(t) \frac{1}{t}\, d Q_\tau.$$

$$= O\left(\frac{1}{Q_n}\right) + o\left[\frac{\epsilon(n)}{\log n}\right] + o\left(\frac{1}{Q_n}\right) \int_{n^{-1}}^{\eta} \frac{\epsilon(s)\, Q_{[s]}}{s \log s}\, ds$$

$$+ o\left(\frac{1}{Q_n}\right) \int_{n^{-1}}^{\eta} \frac{\epsilon(s)}{\log s}\, dQ_{[s]}.$$

$$= o(1) + o\left(\frac{1}{Q_n}\right) \sum_{\nu=2}^{n} \frac{\epsilon(\nu)\, Q_\nu}{\nu \log \nu} + o\left(\frac{1}{Q_n}\right) \int_{n^{-1}}^{\eta} dQ_{[s]}$$

$$= o(1) + o\left(\frac{1}{Q_n}\right) \sum_{\nu=0}^{n} q_\nu$$

$$= o(1)$$

uniformly in E by the hypothesis of the theorem. Hence, we have proved that $I + J = o(1)$. Consequently, the theorem follows.

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