EFFECTS OF COUPLE STRESSES ON THE PROPAGATION OF WAVES IN AN ELASTIC LAYER IMMERSED IN AN INFINITE LIQUID

by D. K. BANERJI *and* P. R. SENGUPTA, *Department of Mathematics, University of Kalyani, Kalyani, Nadia, West Bengal*

(*Received* 26 *July* 1976)

In this paper, the effects of couple stresses for the propagation of liquid-coupled longitudinal waves, flexural waves for a layer have been investigated. The effects of couple stresses for the propagation of liquid-coupled Rayleigh waves and Stoneley waves have also been studied. It has been shown that the effects of couple stresses increase the real part and decrease the imaginary part of the wave speed of the above attenuated waves.

1. INTRODUCTION

In the classical theory of elasticity, it is assumed that the action across any infinitesimal surface element within the solid is equivalent to a force only. It is expected that the force across a hypothetical plane within the solid should be statically equivalent to a force and a couple. Utilizing this idea, Voigt (1887) and Cosserat (1909) originated a theory of deformation of continua by considering the usual force per unit area and with an additional couple per unit area across a surface within a bulk of material or on its boundary. Toupin (1962) derived the associative constitutive equations for finite deformation of perfectly elastic materials. Mindlin and Tiersten (1962) formulated a linearized theory of couple-stress elasticity. Making use of the theory of Mindlin and Tiersten (1962) the effects of couple stresses were studied on surface waves in elastic media and propagation of waves in an elastic layer by Sengupta and Ghosh (1974a, b). The influence of gravity on wave propagation in an elastic layer was investigated by De and Sengupta (1974). In this paper, using the theory of Mindlin and Tiersten (1962), we have investigated the problem of the interaction of underwater explosion waves with an elastic layer. The layer has been taken here as a two-dimensional wave guide.

2. FORMULATION OF THE PROBLEM AND THE BOUNDARY CONDITIONS

Let us introduce a Cartesian frame of reference $0\,x_1x_2x_3$ in the middle plane of the layer. Let x_3-axis be directed downwards and $x_3 = \pm H$ be the boundary planes of the layer, wherein a monochromatic wave propagates with constant velocity c along the x_1-axis.

To obtain the period equation of the problem, it is sufficient to consider the plane waves.

The non-zero displacement components u_1, u_3 at any point of the layer are written in the form

$$u_1 = \Phi_{,1} - \Psi_{,3}\ , \ u_3 = \Phi_{,3} + \Psi_{,1}. \qquad \ldots(1)$$

where $\Phi(x_1, x_3, t)$ and $\Psi(x_1, x_3, t)$ are the displacement potentials.

For linear theory of couple stress elasticity, the displacement equations of motion are (Mindlin and Tiersten 1962)

$$\mu\nabla^2 u_i + (\lambda + \mu)\, u_{j,ji} + \eta\nabla^2(u_{j,ji} - \nabla^2 u_i) = \rho\ddot{u}_i. \qquad \ldots(2)$$
$$(i, j = 1, 2, 3)$$

With the help of eqn. (1), the above fundamental equations (2) reduce to the following differential equations :

$$\left.\begin{aligned} &\nabla^2\Phi - c_1^{-2}\ddot{\Phi} = 0 \\ &\nabla^2\Psi - l^2\nabla^4\Psi - c_2^{-2}\ddot{\Psi} = 0 \end{aligned}\right\} \qquad \ldots(3)$$

where

$$\left.\begin{aligned} &c_1^2 = \frac{\lambda + 2\mu}{\rho}\ , \ c_2^2 = \frac{\mu}{\rho}\ , \ l^2 = \frac{\eta}{\mu}\ , \\ &\nabla^2 = \partial_{,11} + \partial_{,22}. \end{aligned}\right\} \qquad \ldots(4)$$

η is a constant characterizing the effect of the couple stresses in the layer; λ, μ are the Lame's constants; and ρ is the density of the layer.

The following boundary conditions should be satisfied on the edges (at the solid-liquid interface) of the layer $x_3 = \pm H$:

$$\left.\begin{aligned} &\sigma_{33} = \sigma'_{33}\ , \ \sigma_{31} = \mu_{32} = 0 \\ &u_3 = u'_3\ . \end{aligned}\right\} \qquad \ldots(5)$$

The unprimed and primed quantities are related to the solid and liquid respectively.

Due to the participation of the liquid, let us introduce the potentials Φ_0, Φ_2 respectively for the liquid above or below the layer. Then,

$$\left.\begin{aligned} &u'_1 = \Phi_{0,1}\ , \ u'_3 = \Phi_{0,3} \\ \text{and} \quad &u'_1 = \Phi_{2,1}\ , \ u'_3 = \Phi_{2,3} \end{aligned}\right\} \qquad \ldots(6)$$

are respectively the displacements of liquids above and below the layer.

The appropriate solutions of the equations of motion for the liquid are

$$\left.\begin{aligned} &\Phi_0 = A_0 e^{\nu_0 x_3 + i(\omega t - \alpha x_1)} \\ &\Phi_2 = A_2 e^{-\nu_0 x_3 + i(\omega t - \alpha x_1)} \end{aligned}\right\} \qquad \ldots(7)$$

where

$$\nu_0 = (\alpha^2 - k_{\alpha_0}^2)^{1/2},\ k_{\alpha_0}^2 = \frac{\omega^2}{\alpha_0^2},\ \alpha_0^2 = \frac{\lambda_0}{\rho_0}, \qquad \dots(8)$$

$\lambda_0, \rho_0, \alpha_0$ are the constants characterizing the properties of the liquid.

Now, the normal stress σ'_{33} for the liquid is

$$\sigma'_{33} = -\rho_0\omega^2\Phi_0 \quad \text{or} \quad -\rho_0\omega^2\Phi_2 \qquad \dots(9)$$

according as the liquid is above or below the layer. The force stresses σ_{ij} and the couple stresses μ_{ij} for the layer are given by (Mindlin and Tiersten 1962)

$$\left.\begin{aligned} &\sigma_{11} = 2\mu e_{11} + \lambda(e_{11} + e_{33}),\ \sigma_{33} = 2\mu e_{33} + \lambda(e_{11} + e_{33}),\\ &\sigma_{13} + \sigma_{31} = 4\mu e_{13},\ \sigma_{13} - \sigma_{31} = \mu_{12,1} + \mu_{32,3},\\ &\mu_{32} = 4\eta(e_{31,3} - e_{33,1}),\ \mu_{12} = 4\eta(e_{11,3} - e_{13,1}),\\ &e_{ii} = u_{i,i},\ e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),\ i, j = 1, 3). \end{aligned}\right\} \qquad \dots(10)$$

Expressing the required stresses [according to the boundary conditions (5)] for the layer in terms of the displacement potentials, we have

$$\left.\begin{aligned} &\sigma_{33} = 2\mu(\Phi_{,33} + \Psi_{,13}) + \lambda\nabla^2\Phi,\\ &\sigma_{31} = \mu(2\Phi_{,13} - \Psi_{,33} + \Psi_{,11}) + \eta\nabla^4\Psi,\\ &\mu_{32} = -2\eta\nabla^2(\Psi_{,3}). \end{aligned}\right\} \qquad \dots(11)$$

3. Solutions and Phase Velocity Equations

In this section, we shall solve the equations of motion (3) with the help of the appropriate boundary conditions (5) of the problem. Let us assume

$$\left.\begin{aligned} &\Phi = f^*(x_3)e^{i(\omega t - \alpha x_1)},\\ &\Psi = g^*(x_3)e^{i(\omega t - \alpha x_1)}. \end{aligned}\right\} \qquad \dots(12)$$

Substituting (12) in (3) and after a little calculation, we obtain

$$\left.\begin{aligned} &\Phi = (A \sinh \nu_1 x_3 + B \cosh \nu_1 x_3)e^{i(\omega t - \alpha x_1)},\\ &\Psi = (C \sinh \nu_2 x_3 + D \cosh \nu_2 x_3 + E \sinh \nu_3 x_3\\ &\qquad + F \cosh \nu_3 x_3)e^{i(\omega t - \alpha x_1)}, \end{aligned}\right\} \qquad \dots(13)$$

where

$$\left.\begin{aligned} &\nu_1 = (\alpha^2 - k_1^2)^{1/2},\ \nu_2 = (\alpha^2 - p^2)^{1/2},\ \nu_3 = (\alpha^2 + q^2)^{1/2},\\ &p^2 = \frac{1}{2l^2}[(1 + 4k_2^2 l^2)^{1/2} - 1],\ q^2 = \frac{1}{2l^2}[(1 + 4k_2^2 l^2)^{1/2} + 1],\\ &k_1^2 = \frac{\omega^2}{c_1^2},\ k_2^2 = \frac{\omega^2}{c_2^2}. \end{aligned}\right\} \qquad \dots(14)$$

Similarly, as in elasto-kinetics, the general problem of propagation of waves can be reduced to two problems, viz., the consideration of symmetric and antisymmetric vibrations.

Symmetric vibrations — This type of vibrations is characterized by the symmetry of the displacement component u_1 and force stresses σ_{11}, σ_{33} and couple stresses μ_{32} with respect to the plane $x_3 = 0$. In this case, we have to put in the expression (13)

$$A = D = F = 0 \qquad \ldots(15)$$

Expressing the boundary conditions (5) in terms of the functions Φ, Ψ, Φ_0 and Φ_2 by the relations (6), (9) and (11) and finally using the solutions (7) with $A_0 = A_2$, and solutions (13) with the conditions (15), we obtain a system of four homogeneous equations with four unknown constants. Eliminating the unknown constants from this system of equations, we get the following characteristic equation in the form of a transcendental equation for the symmetric mode of vibrations of the layer

$$\left.\begin{aligned} &\alpha^2 (2\alpha^2 - k_2^2)\left(2 - \frac{c^2}{c_2^2}\right)\left(1 + \frac{\nu_2 p^2}{\nu_3 q^2}\frac{\tanh(\nu_3 H)}{\tanh(\nu_2 H)}\right) - 4\alpha^2\nu_1\nu_2 \\ &\times\left(1 + \frac{p^2}{q^2}\right)\frac{\tanh(\nu_1 H)}{\tanh(\nu_2 H)} + \frac{\rho_0\omega^2\nu_1}{\rho\nu_0 c_2^2}\, k_2^2\left(1 + \frac{\nu_2 p^2}{\nu_3 q^2}\frac{\tanh(\nu_3 H)}{\tanh(\nu_2 H)}\right) \\ &\times \tanh \nu_1 H = 0. \end{aligned}\right\} \qquad \ldots(16)$$

Making l tending to zero, eqn. (16) reduces to the classical result (Osborne and Hart 1945) with some change of notations.

Antisymmetric vibrations — Let us now consider the particular case where the displacement component u_1, the stresses σ_{11}, σ_{33} and μ_{32} are antisymmetric with respect to the plane $x_3 = 0$. This postulate will be satisfied, if we assume

$$B = C = E = 0 \qquad \ldots(17)$$

Now, using eqns. (17), (13), (11), (9), (7), (6) and the boundary conditions (5) with $A_0 = - A_2$, we obtain a system of four homogeneous equations with four unknown constants. Making the determinant of this system equal to zero, we arrive at the following phase velocity equation :

$$\left.\begin{aligned} &\alpha^2(2\alpha^2 - K_2^2)\left(2 - \frac{c^2}{c_2^2}\right)\left(1 + \frac{\nu_2 p^2}{\nu_3 q^2}\frac{\tanh(\nu_2 H)}{\tanh(\nu_3 H)}\right)\tanh \nu_1 H \\ &- 4\alpha^2\nu_1\nu_2\left(1 + \frac{p^2}{q^2}\right)\tanh(\nu_2 H) \\ &+ \frac{\rho_0\omega^2\nu_1}{\rho\nu_0 c_2^2}\, k_2^2\left(1 + \frac{\nu_2 p^2}{\nu_3 q^2}\frac{\tanh(\nu_2 H)}{\tanh(\nu_3 H)}\right) = 0. \end{aligned}\right\} \qquad \ldots(18)$$

If the couple-stress parameter $l \to 0$, then with some change of notations, we get the classical result (Osborne and Hart 1945).

4. Discussion

It is interesting to discuss the period equations (16) and (18) in the following cases.

(A) *Large wavelength* ($c > \alpha_0$)

(i) If the length of the wave is large compared to the thickness of the layer, $2H$, the quantities $\nu_1 H$, $\nu_2 H$ and αH are small, but $\nu_3 H$ is not small. Replacing the hyperbolic tangents by their arguments, we have from (16)

$$\left.\begin{aligned}&\left(2-\frac{c^2}{c_2^2}\right)(2\alpha^2-k_2^2)\left(1+\frac{p^2}{q^2}\cdot\tanh\frac{\nu_3 H}{\nu_3 H}\right)-4\nu_1^2\left(1+\frac{p^2}{q^2}\right)\\&+\frac{\rho_0\omega^2\nu_1^2 H}{\rho\nu_0}\cdot\frac{c^4}{c_2^4}\left(1+\frac{p^2}{q^2}\cdot\tanh\frac{\nu_3 H}{\nu_3 H}\right)=0.\end{aligned}\right\} \quad ...(19)$$

Equation (19) will determine the velocity of long longitudinal waves of the layer with the effect of the liquid and under the influence of couple stress. Since the couple stress parameter l is small, we get from (19), the following approximate form

$$\left(2-\frac{c^2}{c_2^2}\right)^2-4\left(1-\frac{c^2}{c_1^2}\right)\left(1+\frac{c^2}{c_2^2}\cdot\alpha^2 l^2\right)$$

$$+\frac{\rho_0\alpha H c^4}{\rho c_2^4}\cdot\frac{\left(1-\dfrac{c^2}{c_1^2}\right)}{\left(1-\dfrac{c^2}{\alpha_0^2}\right)^{1/2}}=0 \quad ...(20)$$

wherefrom by assuming the effect of the liquid to be small, we obtain the following approximate equation

$$\frac{c}{c_2}=D^{1/2}\left[1+\frac{i}{2}\frac{\rho_0\alpha_0}{\rho c_2^2}\cdot\omega H\left(\frac{1}{D}-\frac{c_2^2}{c_1^2}\right)\left(1-\frac{4\alpha^2 l^2 c_2^2}{c_1^2}\right)\right], \quad ...(21)$$

where

$$D=4\left(1-\frac{c_2^2}{c_1^2}+\alpha^2 l^2\right)\left(1-\frac{4\alpha^2 l^2 c_2^2}{c_1^2}\right).$$

Let us now assume that the elastic solid satisfies the Poisson's condition $\lambda=\mu$.

Then, (21) reduces to

$$\frac{c}{c_2}=2\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(1+\frac{\alpha^2 l^2}{6}\right)^{\frac{1}{2}}\left[1+\frac{i}{2}\frac{\rho_0\alpha_0}{\rho c_2^2}\,\omega H\right.$$

$$\left.\left(\frac{3}{8\left(1+\dfrac{\alpha^2 l^2}{6}\right)}-\frac{1}{8}\right)\left(\frac{3-4\alpha^2 l^2}{3}\right)\right]. \quad ...(22)$$

The first part of eqn. (22) represents the expression for long longitudinal vibrations of the layer and the second part represents the attenuating effect of the liquid under the influence of couple stress.

Now, if $l \to 0$, we obtain from (22), the classical result (Osborne and Hart 1945) with $\lambda = \mu$.

$$\frac{c}{c_2} = 2\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(1 + \frac{i}{24} \cdot \frac{\rho_0 \alpha_0}{\rho c_2^2} \cdot \omega H\right). \qquad \text{...(23)}$$

From (22) and (23), it is easy to conclude that the presence of the couple stress increases the real part of the phase velocity for long longitudinal vibrations of the layer and diminishes the attenuating effect of the liquid for a fixed circular frequency (low).

(ii) For long waves, as compared to the thickness $2H$ of the layer, the quantities $\nu_1 H$, $\nu_2 H$ and αH can be regarded as small. Retaining the terms up to the third order in the expansion of hyperbolic tangents, eqn. (18) reduces to

$$4\nu_2^2\left(1 + \frac{p^2}{q^2}\right)(1 - \tfrac{1}{3}\nu_2^2 H^2) + (2\alpha^2 - k_2^2)\left(\frac{c^2}{c_2^2} - 2\right)\left(1 - \frac{\nu_1^2 H^2}{3}\right)$$
$$\times\left(1 + \frac{\nu_2^2 p^2 H}{\nu_3 q^2}\, \frac{(1 - \frac{1}{3}\nu_2^2 H^2)}{\tanh(\nu_3 H)}\right) - \frac{\rho_0 c^2 \omega^2}{\rho c_2^4 \nu_0 H}$$
$$\times\left(1 + \frac{\nu_2^2 p^2 H}{\nu_3 q^2} \cdot \frac{(1 - \frac{1}{3}\nu_2^2 H^2)}{\tanh(\nu_3 H)}\right) = 0. \qquad \text{...(24)}$$

Assuming l to be small and performing some algebraic manipulation of eqn. (24), we derive the phase velocity of liquid-coupled long flexural waves of the layer under the influence of couple stress in the following form :

$$\frac{c^2}{c_2^2} = \frac{\left[\frac{4\alpha^2 H^2}{3}\left(1 - \frac{c_2^2}{c_1^2}\right) + 4\alpha^2 l^2\right] \cdot \frac{\rho \alpha H}{\rho_0}}{\left[1 + \frac{\rho}{\rho_0}\alpha H\right]}. \qquad \text{...(25)}$$

Now, making l tending to zero, eqn. (25) becomes

$$\frac{c^2}{c_2^2} = \frac{\frac{4}{3}\alpha^3 H^3\left(1 - \frac{c_2^2}{c_1^2}\right)\frac{\rho}{\rho_0}}{\left(1 + \frac{\rho}{\rho_0}\alpha H\right)}. \qquad \text{...(26)}$$

This is in fair agreement with the corresponding classical result (Osborne and Hart 1945).

From (25) and (26), it is concluded that with increase in the couple stress parameter l, the phase velocity for long flexural waves of the layer also increases without any attenuating effect of the liquid.

(B) *Small wavelength* ($c > \alpha_0$ and $c < \alpha_0$)

(i) Firstly, let us consider the case $c > \alpha_0$. When the length of the wave is very small compared to the thickness of the layer, i.e., the problem approaches the case of a semi-infinite medium, then $\nu_1 H$, $\nu_2 H$, $\nu_3 H$ and αH are large. In this case, the determinants for symmetric and antisymmetric motions approach one and the same limiting form

$$(2\alpha^2 - K_2^2)\left(2 - \frac{c^2}{c_2^2}\right)\left(1 + \frac{\nu_2 p^2}{\nu_3 q^2}\right) - 4\,\nu_1\nu_2\left(1 + \frac{p^2}{q^2}\right) + \frac{\rho_0 c^2 \nu_1}{\rho \nu_0 c_2^2} \cdot K_2^2\left(1 + \frac{\nu_2 p^2}{\nu_3 q^2}\right) = 0 \qquad \ldots(27)$$

The couple stress parameter l being small, we get the following approximate form of eqn. (27)

$$\left(2 - \frac{c^2}{c_2^2}\right)^2 - 4\left(1 - \frac{c^2}{c_1^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{c_2^2} + \frac{c^4}{c_2^4}\cdot \alpha^2 l^2\right)^{\frac{1}{2}} \times\left(1 + \frac{c^2}{c_2^2}\alpha^2 l^2\right) + \frac{\rho_0 c^4}{\rho c_2^4}\,\frac{\left(1 - \frac{c^2}{c_1^2}\right)^{\frac{1}{2}}}{\left(1 - \frac{c^2}{\alpha_0^2}\right)^{\frac{1}{2}}} = 0 \qquad \ldots(28)$$

The first two terms of eqn. (28) represent the expression for Rayleigh waves propagated along the free surface of a semi-infinite solid medium under the influence of couple stress (Sengupta and Ghosh 1974a). The third term represents the effect produced by the liquid Therefore, eqn. (28) corresponds to the phase velocity equation of Rayleigh waves on the floor of the ocean with due consideration for couple stress.

Let us assume the water correction term to be small. Substituting $c = R(1 + \epsilon)$, in Eqn. (28), we may write with sufficient approximation (i.e., retaining only the first power of ϵ)

$$\epsilon = \left[\frac{\frac{\rho_0}{\rho}\frac{R^4}{c_2^4}\left(1 - \frac{R^2}{c_1^2}\right)^{\frac{1}{2}}}{4\left(1 - \frac{R^2}{\alpha_0^2}\right)^{\frac{1}{2}}}\right] \cdot \left\{\frac{R^2}{c_1^2} \cdot \left[\frac{\left(1 - \frac{R^2}{c_2^2} + \frac{R^4}{c_2^4}\alpha^2 l^2\right)}{\left(1 - \frac{R^2}{c_1^2}\right)}\right] \times\left(1 + \frac{R^2}{c_2^2}\alpha^2 l^2\right) + \frac{R^2}{c_2^2}\left(1 + \frac{R^2}{c_2^2}\alpha^2 l^2\right) \times\left[\frac{\left(1 - \frac{R^2}{c_1^2}\right)}{\left(1 - \frac{R^2}{c_2^2} + \frac{R^4}{c_2^4}\alpha^2 l^2\right)}\right]\left(1 - \frac{2R^2}{c_2^2}\alpha^2 l^2\right) - \right.$$

(*equation continued on p.* 24)

$$- \frac{R^2}{c_2^2}\left(2 - \frac{R^2}{c_2^2}\right) - 2\frac{R^2}{c_2^2}\alpha^2 l^2 \left(1 - \frac{R^2}{c_1^2}\right)^{\frac{1}{2}}$$

$$\times \left(1 - \frac{R^2}{c_2^2} + \frac{R^4}{c_2^4}\alpha^2 l^2\right)^{\frac{1}{2}} \Bigg\}^{-1} \quad \ldots(29)$$

where R is the velocity of Rayleigh waves in a half-space under the influence of couple stress. It has been seen (Sengupta and Ghosh 1974a) that the values of $\alpha^2 l^2$ must be less or equal to 0.09 in order to satisfy all the conditions for the propagation of Rayleigh waves under the influence of couple stress. Taking this restriction into account, we perform the following numerical calculations.

Numerical Calculations

To study the phase velocity equation (28) corresponding to different values of the couple stress parameter l, we compile Table I to get a clear picture of the variation of phase velocity with the attenuation of the Rayleigh waves. For this, we consider the variation of $\alpha^2 l^2$ in eqn. (29) and compute the various values of ϵ corresponding to the various values of R for a particular layer.

In computing the numerical results, the following assumptions are made :

For the layer

$$\lambda = \mu,\ \rho = 7.6 \text{ g/cc},\ c_2 = 0.33 \times 10^6 \text{ cm/sec}$$

and for the liquid

$$\rho_0 = 1 \text{ g/cc},\ \alpha_0 = 0.15 \times 10^6 \text{ cm/sec.} \quad \ldots(30)$$

Table I

$\alpha^2 l^2$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
R, cm/sec	306619.5	304550.4	304256.7	303405.6	303402.8
ϵ	$i1.098 \times 10^{-2}$	$i1.124 \times 10^{-2}$	$i1.129 \times 10^{-2}$	$i1.16069 \times 10^{-2}$	$i1.16072 \times 10^{-2}$

From Table I, it is interesting to observe that with increase in the parameter $\alpha^2 l^2$, the real part of the phase velocity, the root of eqn. (28), increases and the imaginary part decreases, i.e., with the cumulative of the parameter $\alpha^2 l^2$, the velocity of the Rayleigh waves increases, but the attenuation of the Rayleigh waves decreases.

(ii) Secondly, let us now consider the case $c < \alpha_0$. For very short wavelengths, i.e., for small waves in comparison to the thickness of layer, we have αH, $\nu_1 H$, $\nu_2 H$, $\nu_3 H$ to be large and in this case, the symmetric and antisymmetric forms of the phase velocity equations approach the same limiting form. If c is assumed to be real and less than α_0, the equation which must be satisfied by c is

$$\left(1-\frac{c^2}{\alpha_0^2}\right)^{\frac{1}{2}}\left[\left(2-\frac{c^2}{c_2^2}\right)^2-4\left(1-\frac{c^2}{c_1^2}\right)^{\frac{1}{2}}\left(1-\frac{c^2}{c_2^2}+\frac{c^4}{c_2^4}\cdot\alpha^2l^2\right)^{\frac{1}{2}}\right.$$
$$\left.\times\left(1+\frac{c^2}{c_2^2}\cdot\alpha^2l^2\right)\right]+\frac{\rho_0c^4}{\rho c_2^4}\cdot\left(1-\frac{c^2}{c_1^2}\right)^{\frac{1}{2}}=0. \qquad \ldots(31)$$

These two terms are of opposite signs, so that if $c < \alpha_0$, there is a possibility of a real root of eqn. (31). From the above equation, we get the following approximation

$$\frac{c^2}{\alpha_0^2}=1-\left[\frac{\frac{\rho_0}{\rho}\cdot\frac{\alpha_0^4}{c_2^4}\left(1-\frac{\alpha_0^2}{c_1^2}\right)^{\frac{1}{2}}}{\left(2-\frac{\alpha_0^2}{c_2^2}\right)^2-4\left(1-\frac{\alpha_0^2}{c_1^2}\right)^{\frac{1}{2}}\left(1-\frac{\alpha_0^2}{c_2^2}+\frac{\alpha_0^4}{c_2^4}\alpha^2l^2\right)^{\frac{1}{2}}\left(1+\frac{\alpha_0^2}{c_2^2}\alpha^2l^2\right)}\right]^2. \qquad \ldots(32)$$

Equation (32) is useful for determining the velocity of Stonely waves under the influence of couple stress in the liquid-solid interface.

Numerical Calculations

In Table II, the values of c/α_0 have been computed by taking the material constants of a particular elastic solid layer and the liquid as exhibited in (30). For different values of α^2l^2, the values of c/α_0 are computed from eqn. (32).

Table II

α^2l^2	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
c/α_0	0.9997218	0.9997034	0.9997015	0.9997013	0.9997013

From Table II, we note that the velocity of Stoneley waves under the influence of couple stress at the liquid-solid interface increases (though the increase is small) with increase in the parameter α^2l^2.

References

Cosserat, E., and Cosserat, F. (1909). Theory des corps deformables. A. Herman et Fils, Paris.

De, S. N., and Sengupta, P. R. (1974). *J. acoust. Soc Am.*, **55** (5), 919-21.

Mindlin, R. D., and Tiersten, H. F. (1962). Effects of couple stresses in linear elasticity, *Arch. Rat. Mech. Analysis*, **11**, 415.

Osborne, M. F., and Hart, S. D. (1945). Transmission, reflection and guiding of an exponential pulse by a steel plate in water—I. *J. acoust. Soc. Am.*, **17**, 1-18.

Sengupta, P. R., and Ghosh, B. (1974a). Effect of couple stresses on the surface waves in elastic media. *Gerlands Beitr. Geophysik, Leipzig*, **83**, 309-18.

Sengupta, P. R., and Ghosh, B. (1974b). Effect of couple stresses on the propagation of waves in an elastic layer. *Pure appl. Geophys.*, **112**, 331-38.

Toupin, R. A. (1962). Theories of elasticity with couple stresses. *Arch. Rat. Mech. Analysis*, **17**, 85.

Voigt, W. (1887). Theoretische studien über die Elasticitäts-Varhättnisse der Krystalle—I, II. *Abh. Königh. Ges. derwiss., Gottingen*, 34.