COMMON FIXED POINTS ON COMPLETE AND COMPACT METRIC SPACES

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It is proved that if S and T are two continuous mappings of the complete metric space X into itself such that

$$\rho(S^2x, T^2y) \leqslant c \max \{\rho(Sx, Ty), \rho(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$, then S and T have a unique common fixed point z.

The following theorem was proved recently by Ray (1976) :

Theorem 1 — If S and T are two mappings of the metric space X into itself such that

$$\rho(Sx, Ty) \leqslant c\rho(x, y)$$

for all x, y in X, where $0 \leqslant c < 1$ and if for some x_0 in X the sequence $\{x_n\}$ consisting of the points

$$x_{2n+1} = Sx_{2n}, \ x_{2(n+1)} = Tx_{2n+1}, \ n = 0, 1, 2, \ldots,$$

has a subsequence $\{x_{n(k)}\}$ convergent to a point z in X, then S and T have the unique common fixed point z.

This theorem follows immediately from the following theorem which was proved by Fisher (1977).

Theorem 2 — If S and T are two mappings of the metric space X into itself such that

$$\rho(Sx, Ty) \leqslant c\rho(x, y)$$

for all x, y in X, where $0 \leqslant c < 1$, then S and T are identical contraction mappings.

We now prove a theorem for two mappings S and T on a metric space which are not necessarily equal.

Theorem 3 — If S and T are two continuous mappings of the complete metric space X into itself such that

$$\rho(S^2x, T^2y) \leqslant c \max \{\rho(Sx, Ty), \rho(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$, then S and T have a unique common fixed point z.

PROOF : Let x be an arbitrary point in X. Then

$$\begin{aligned}\rho(S^{2n}x, T^{2n+1}x) &\leqslant c \max \{\rho(S^{2n-1}x, T^{2n}x), \ \rho(S^{2n-2}x, T^{2n-1}x)\} \\ &\leqslant c \max \{c\rho(S^{2n-2}x, T^{2n-1}x), c\rho(S^{2n-3}x, T^{2n-2}x), \\ &\qquad \rho(S^{2n-2}x, T^{2n-1}x)\} \\ &= c \max \{c\rho(S^{2n-3}x, T^{2n-2}x), \rho(S^{2n-2}x, T^{2n-1}x)\} \\ &\leqslant c^2 \max \{\rho(S^{2n-3}x, T^{2n-2}x), \rho(S^{2n-4}x, T^{2n-3}x)\} \\ &\leqslant c^n \max \{\rho(Sx, T^2x), \rho(x, Tx)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\rho(T^{2n+1}x, S^{2n+2}x) &\leqslant c \max \{\rho(T^{2n}x, S^{2n+1}x), \rho(T^{2n-1}x, S^{2n}x)\} \\ &\leqslant c^n \max \{\rho(T^2x, S^3x), \rho(Tx, S^2x)\}.\end{aligned}$$

Putting

$$M = \max \{\rho(x, Tx), \rho(Tx, S^2x), \rho(Sx, T^2x), \rho(T^2x, S^3x)\}$$

we have

$$\begin{aligned}&\rho(S^{2n}x, S^{2n+2r}x) \\ &\quad \leqslant \rho(S^{2n}x, T^{2n+1}x) + \rho(T^{2n+1}x, S^{2n+2}x) + \dots + \rho(T^{2n+2r-1}x, S^{2n+2r}x) \\ &\quad \leqslant 2M(c^n + c^{n+1} + \dots) \\ &\quad = \frac{2Mc^n}{1-c}\end{aligned}$$

$$\begin{aligned}\rho(S^{2n}x, T^{2n+2r+1}x) &\leqslant \rho(S^{2n}x, T^{2n+1}x) + \dots + \rho(S^{2n+2r}x, T^{2n+2r+1}x) \\ &\leqslant \frac{2Mc^n}{1-c}\end{aligned}$$

$$\begin{aligned}\rho(T^{2n+1}x, T^{2n+2r+1}x) &\leqslant \rho(T^{2n+1}x, S^{2n+2}x) + \dots + \rho(S^{2n+2r}x, T^{2n+2r+1}x) \\ &\leqslant M(c^n + 2c^{n+1} + \dots) \\ &\leqslant \frac{2Mc^n}{1-c}\end{aligned}$$

and

$$\begin{aligned}\rho(T^{2n+1}x, S^{2n+2r}x) &\leqslant \rho(T^{2n+1}x, S^{2n+2}x) + \dots + \rho(T^{2n+2r-1}x, S^{2n+2r}x) \\ &\leqslant \frac{2Mc^n}{1-c}.\end{aligned}$$

Since $c < 1$, it follows that the sequence

$$\{x, Tx, S^2x, \ldots, T^{2n-1}x, S^{2n}x, T^{2n+1}x, \ldots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X.

We can prove similarly that the sequence

$$\{x, Sx, T^2x, \ldots, S^{2n-1}x, T^{2n}x, S^{2n+1}x, \ldots\}$$

is also a Cauchy sequence and so has a limit z' in X. We thus have

$$\lim_{n\to\infty} S^{2n}x = \lim_{n\to\infty} T^{2n+1}x = z, \quad \lim_{n\to\infty} T^{2n}x = \lim_{n\to\infty} S^{2n+1}x = z'.$$

Further,

$$\rho(S^{2n}x, T^{2n}x) \leqslant c \max\{\rho(S^{2n-1}x, T^{2n-1}x), \rho(S^{2n-2}x, T^{2n-2}x)\}$$

and so on letting n tend to infinity, we have

$$\begin{aligned}\rho(z, z') &\leqslant c \max\{\rho(z', z), \rho(z, z')\}\\ &= c\rho(z, z').\end{aligned}$$

Since $c < 1$, it follows that $z = z'$ and so

$$\lim_{n\to\infty} S^n x = \lim_{n\to\infty} T^n x = z.$$

Using the inequality of S and T, we now see that

$$Sz = Tz = z$$

so that z is a common fixed point of S and T.

Now suppose w is a second common fixed point of S and T. Then,

$$\begin{aligned}\rho(z, w) &= \rho(S^2z, T^2w)\\ &\leqslant c \max\{\rho(Sz, Tw), \rho(z, w)\}\\ &= c\rho(z, w).\end{aligned}$$

It follows that $z = w$ and so the common fixed point z is unique. This completes the proof of the theorem.

We now show that the two mappings S and T are not necessaiily equal. To do this, let X be the set of real numbers x with $0 \leqslant x \leqslant 1$. Define a metric by

$$\rho(x, y) = |x - y|$$

for all x, y in X, so that X is complete. Define continuous mappings S and T on X by

$$Sx = \tfrac{1}{2}x, \; Tx = 0$$

for all x in X. It is easily shown that

$$\rho(S^2x, T^2y) \leqslant \tfrac{1}{2} \max \{\rho(Sx, Ty), \rho(x, y)\}$$

for all x, y in X. All the conditions of the theorem are, therefore, satisfied, but S and T are not identical.

We note that the condition that S and T both be continuous is also essential. To see this, let X be the metric space defined in the previous example and also let S be the same continuous mapping. We will define a discontinuous mapping T on X by

$$T(0) = 1$$

and

$$Tx = \tfrac{1}{2}x$$

for all other x in X. We again have

$$\rho(S^2x, T^2y) \leqslant \tfrac{1}{2} \max \{\rho(Sx, Ty), \rho(x, y)\}$$

for all x, y in X, but S and T have no common fixed points.

By noting that

$$b\rho(Sx, Ty) + c\rho(x, y) \leqslant \max \{\rho(Sx, Ty), \rho(x, y)\}$$

where $0 \leqslant b, c, b + c \leqslant 1$, we prove the following theorem.

Theorem 4 — If S and T are two continuous mappings of the complete metric space X into itself such that

$$\rho(S^2x, T^2y) \leqslant b\rho(Sx, Ty) + c\rho(x, y)$$

for all x, y in X, where $0 \leqslant b, c, b + c < 1$, then S and T have a unique common fixed point z.

We now prove the following theorem for compact metric spaces.

Theorem 5 — If T is a continuous mapping of the compact metric space X into itself such that for all x, y in X, either

$$\rho(T^2x, T^2y) < \max \{\rho(Tx, Ty), \rho(x, y)\}$$

if $\max \{\rho(Tx, Ty), \rho(x, y)\} > 0$, or

$$\rho(T^2x, T^2y) = 0$$

if $\rho(Tx, Ty) = \rho(x, y) = 0$, then T has a unique fixed point z.

Proof : Define a function f on X by

$$f(x) = \max \{\rho(Tx, T^2x), \rho(x, Tx)\}$$

for all x in X. Since ρ and T are continuous functions, it follows that f is a continuous function. Since X is compact, there exists a point y in X such that

$$f(y) = \inf \{f(x) : x \in X\}.$$

The assumption that

$$f(y) = \max \{\rho(Ty, T^2y), \rho(y, Ty)\} \neq 0$$

implies that

$$\rho(T^2y, T^3y) < f(y)$$

and the additional assumption that

$$f(Ty) = \max \{\rho(T^2y, T^3y), \rho(Ty, T^2y)\} \neq 0$$

implies that

$$\begin{aligned}\rho(T^3y, T^4y) &< f(Ty) \\ &< \max \{\rho(Ty, T^2y), \rho(y, Ty)\} \\ &= f(y).\end{aligned}$$

It follows that

$$\begin{aligned}f(T^2y) &= \max \{\rho(T^3y, T^4y), \rho(T^2y, T^3y)\} \\ &< f(y),\end{aligned}$$

contradicting the definition of y. We must, therefore, have either

$$\max \{\rho(Ty, T^2y), \rho(y, Ty)\} = 0$$

or

$$\max \{\rho(T^2y, T^3y), \rho(Ty, T^2y)\} = 0.$$

Either of these cases implies that

$$\rho(Ty, T^2y) = 0$$

and so $Ty = z$ is a fixed point of T. This completes the proof of the theorem.

We now use this theorem to prove the following theorem for compact metric spaces :

Theorem 6 — If S and T are two continuous mappings of the compact metric space X into itself such that for all x, y in X, either

$$\rho(S^2x, T^2y) < \max \{\rho(Sx, Ty), \rho(x, y)\}$$

if $\qquad \max \{\rho(Sx, Ty), \rho(x, y)\} > 0,$

or $\qquad \rho(S^2x, T^2y) = 0$

if $\rho(Sx, Ty) = \rho(x, y) = 0$, then S and T have a unique common fixed point z.

PROOF : If there exists c, with $0 \leqslant c < 1$, such that

$$\rho(S^2x, T^2y) \leqslant c \max \{\rho(Sx, Ty), \rho(x, y)\}$$

for all x, y in X, then the result follows from Theorem 3.

If no such c exists, let $\{c_n\}$ be a monotonically increasing sequence of positive numbers such that $\lim_{n\to\infty} c_n = 1$. Then, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\rho(S^2x_n, T^2y_n) \geqslant c_n \max \{\rho(Sx_n, Ty_n), \rho(x_n, y_n)\}$$

for $n = 1, 2, \dots$. X is compact and so we can find convergent subsequences $\{x_{n(k)}\}$ and $\{y_{n(k)}\}$ converging to x and y respectively. We then have

$$\rho(S^2x_{n(k)}, T^2y_{n(k)}) \geqslant c_{n(k)} \max \{\rho(Sx_{n(k)}, Ty_{n(k)}), \rho(x_{n(k)}, y_{n(k)})\}$$

and on letting k tend to infinity, we see that

$$\rho(S^2x, T^2y) \geqslant \max \{\rho(Sx, Ty), \rho(x, y)\}$$

since S and T are continuous. This implies that

$$\rho(S^2x, T^2y) = \rho(Sx, Ty) = \rho(x, y) = 0.$$

We, therefore, have $x = y$ and then

$$Sx = Tx, \quad S^2x = T^2x.$$

We will now prove that $S^nx = T^nx$ for $n = 1, 2, \dots$. Assuming that

$$S^{n-1}x = T^{n-1}x, \quad S^nx \; T^nx$$

for some n, we have

$$\max \{\rho(S^nx, T^nx), \rho(S^{n-1}x, T^{n-1}x)\} = 0$$

and it follows that

$$S^{n+1}x = T^{n+1}x.$$

The result now follows by induction.

We will now put

$$X_1 = \{x, Tx, T^2x, \dots, T^nx, \dots\}.$$

Then $\bar{X}_1$, the closure of X_1, is compact, being a closed subspace of the compact metric space X.

Now let $\bar{x}$ be an arbitrary point in $\bar{X}_1$. Then

$$\bar{x} = \lim_{k\to\infty} T^{n(k)}x$$

for some sequence $\{T^{n(k)}x\}$ in X_1. Thus,

$$T\bar{x} = \lim_{k\to\infty} T^{n(k)+1}x$$

since T is continuous and so $T\bar{x}$ is in $\bar{X}_1$. T is, therefore, a mapping of $\bar{X}_1$ into itself.

Further

$$T\bar{x} = \lim_{k\to\infty} T^{n(k)+1}x = \lim_{k\to\infty} S^{n(k)+1}x = S\bar{x}$$

since $S = T$ on X_1 and so $S = T$ on $\bar{X}_1$.

It follows that T satisfies the conditions of Theorem 5 on the compact metric space $\bar{X}_1$ and so T has a fixed point z in $\bar{X}_1$. Since $S = T$ on $\bar{X}_1$, it follows that z is also a fixed point of S.

Now suppose that S and T have a second distinct common fixed point w. Then

$$\begin{aligned}\rho(z, w) &= \rho(S^2z, T^2w)\\ &< \max\{\rho(Sz, Tw), \rho(z, w)\}\\ &= \rho(z, w)\end{aligned}$$

giving a contradiction. The common fixed point z must, therefore, be unique.

The proof of the final theorem follows.

Theorem 7 — If S and T are two continuous mappings of the compact metric space X into itself such that for all x, y in X, either

$$\rho(S^2x, T^2y) < c\rho(Sx, Ty) + (1 - c)\,\rho(x, y)$$

if $\quad c\rho(Sx, Ty) + (1 - c)\,\rho(x, y) > 0$, or

$$\rho(S^2x, T^2y) = 0$$

if $\rho(Sx, Ty) = \rho(x, y) = 0$, where $0 \leqslant c \leqslant 1$, then S and T have a unique common fixed point z.

References

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