AN APPELL CROSS-SEQUENCE SUGGESTED BY HERMITE POLYNOMIALS

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The Appell cross-sequence relative to the invertible shift invariant operator $e^{-\lambda D^m}$, where λ is any real parameter and m is a positive integer, is studied by the use of finite operator calculus developed by Rota (1973). For $m = 2$ and $\lambda = -\frac{1}{2}$, it becomes the set of ordinary Hermite polynomials. A number of properties, such as summation formulae, the binormal identity, a generating relation, the umbral composition formula and pure recurrence relations are established.

1. INTRODUCTION

In a recent study, Rota (1973) developed in a systematic and rigorous way, a finite operator calculus and employed it to study several well-known polynomial sets. One such polynomial set discussed by him is the Appell set $H_n^{(\nu)}(x)$ relative to the Weirstrass operator defined by

$$W_\nu p(x) = \frac{1}{(2\pi\nu)^{1/2}} \int_{-\infty}^{\infty} e^{-t^2/2\nu}\, p(x+t)\, dt$$

for all polynomials $p(x)$ in the algebra P of polynomials over a field of characteristic zero. He called $H_n^{(\nu)}(x)$ as the Hermite polynomials with variance ν, which reduce to classical Hermite polynomials when $\nu = 1$. In this paper, we study a much more general Appell set (in fact, an Appell cross-sequence) relative to the invertible shift invariant operator $S_{\lambda,m} = e^{-\lambda D^m}$. For $\lambda = -\nu/2$ and $m = 2$, this reduces to the set $H_n^{(\nu)}(x)$. These polynomials or their special cases have been considered earlier by some authors (Gould and Hopper 1962, Lahiri 1971, Srivastava 1976, Rekha 1976 and Saha 1974) either by formal operational methods or by methods of generating functions. As done by Rota (1973), we base our treatment on finite operator calculus. Throughout, we shall use the notations and terminology of Rota (1973).

2. Polynomials $P^{(\lambda)}_{n,m}(x)$ as a Sheffer Set

We define $P^{(\lambda)}_{n,m}(x)$ to be the Appell set whose operator is $S_{\lambda,m} = e^{-\lambda D^m}$, where $D \equiv \frac{d}{dx}$. The ordinary Hermite polynomials correspond to $\lambda = -\frac{1}{2}$ and $m = 2$.

The basic polynomials of the delta operator in this case are $q_n(x) = x^n$. Since $P^{(\lambda)}_{n,m}(x)$ is a Sheffer set relative to the shift invariant operator $S_{\lambda,m}$, we have

$$P^{(\lambda)}_{n,m}(x) = e^{\lambda D^m} x^n$$

$$= \sum_{k=0}^{[n/m]} \frac{n!\, \lambda^k x^{n-mk}}{(n-mk)!\,k!}. \qquad \text{...(2.1)}$$

Evidently, $DP^{(\lambda)}_{n,m}(x) = n\, P^{(\lambda)}_{n-1,m}(x)$, which, in turn, implies that

$$D^k P^{(\lambda)}_{n,m}(x) = (n)^{(k)} P^{(\lambda)}_{n-k,m}(x);\ (n)^{(k)} = n(n-1)\ldots(n-k+1). \qquad \text{...(2.2)}$$

The binomial identity for the polynomials $P^{(\lambda)}_{n,m}(x)$ is

$$P^{(\lambda)}_{n,m}(x+y) = \sum_{k\geqslant 0} \binom{n}{k} y^{n-k} P^{(\lambda)}_{k,m}(x). \qquad \text{...(2.3)}$$

Moreover, if we put $y = 0$ and interchange x and y, we get

$$P^{(\lambda)}_{n,m}(x) = \sum_{k\geqslant 0} \binom{n}{k} P^{(\lambda)}_{k,m}(0)\, x^{n-k}$$

with the help of which we can determine $P^{(\lambda)}_{n,m}(x)$ in terms of their constant terms.

Now, as $P^{(\lambda)}_{n,m}(x)$ is a Sheffer set, its generating function relation is given by (Rota 1973, §5)

$$\sum_{n\geqslant 0} \frac{P^{(\lambda)}_{n,m}(x)}{n!} t^n = \frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} \qquad \text{...(2.4)}$$

where $s(t) = e^{-\lambda t^m}$ is the indicator of the invertible shift invariant operator $S = e^{-\lambda D^m}$ and $q^{-1}(t)$ is the formal power series inverse to $q(t)$, the indicator of the delta operator. Evidently, here $q(t) = t$, so that (2.4) becomes

$$\sum_{n\geqslant 0} \frac{P^{(\lambda)}_{n,m}(x)}{n!} t^n = e^{xt+\lambda t^m}. \quad \text{...(2.5)}$$

For $\lambda = -\nu/2$ and $m = 2$, we get the generating function (Rota 1973, §10) for the Hermite polynomials $H^{(\nu)}_n(x)$ of variance ν.

3. $P^{(\lambda)}_{n,m}(x)$ AS A CROSS-SEQUENCE

It is easily seen that $P^{-\lambda} = e^{\lambda D^m}$ form a one-parameter group of shift invariant operators and for the sequence $q_n(x) = x^n$ (which is of binomial type), the relation

$$P^{(\lambda)}_{n,m}(x) = P^{-\lambda} x^n \quad \text{...(3.1)}$$

holds, from which it follows that $P^{(\lambda)}_{n,m}(x) = P^{[\lambda]}_{n,m}(x)$ is a cross-sequence. Consequently, we have

$$P^{[\lambda+\mu]}_{n,m}(x+y) = \sum_{k=0}^{n} \binom{n}{k} P^{[\lambda]}_{k,m}(x)\, P^{[\mu]}_{n-k,m}(y) \quad \text{...(3.2)}$$

for all λ and μ in the field and for any x and y. If we set $\lambda = \mu$ in (3.2), we get

$$\sum_{k=0}^{n} \binom{n}{k} P^{[\lambda]}_{k,m}(x)\, P^{[\lambda]}_{n-k,m}(y) = 2^{n/m}\, P^{[\lambda]}_{n,m}\left(\frac{x+y}{2^{1/m}}\right). \quad \text{...(3.3)}$$

For $\lambda = -\nu/2$ and $m = 2$, we get the following identity for $H^{(\nu)}_n(x)$:

$$\sum_{k=0}^{n} \binom{n}{k} H^{(\nu)}_k(x)\, H^{(\nu)}_{n-k}(y) = 2^{n/m}\, H^{(\nu)}_n\left(\frac{x+y}{\sqrt{2}}\right). \quad \text{...(3.4)}$$

Putting $\lambda = -\mu$ in (3.2), we get

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} P^{[\lambda]}_{k,m}(x)\, P^{[-\lambda]}_{n-k,m}(y). \quad \text{...(3.5)}$$

when $y = 0$, (3.5) gives

$$x^n = \sum_{k=0}^{[n/m]} \frac{n!\,(-\lambda)^k}{k!\,(n-mk)!} P^{[\lambda]}_{n-mk,m}(x). \qquad ...(3.6)$$

4. Umbral Composition and Generators

(a) As $P^{(\lambda)}_{n,m}(x)$ is an Appell cross-sequence, the umbral composition formula for these polynomials is

$$P^{[\lambda]}_{n,m}(P^{[\mu]}(x)) = P^{[\mu]}_{n,m}(P^{[\lambda]}(x)) = P^{[\lambda+\mu]}_{n,m}(x).$$

Moreover, $P^{[-\lambda]}_{n,m}(x)$ is the inverse set of $P^{[\lambda]}_{n,m}(x)$ and hence

$$P^{[\lambda]}_{n,m}(P^{[-\lambda]}(x)) = x^n.$$

(b) *Generator of the sequence* $P^{[\lambda]}_{n,m}(x)$ — Since $P^{-\lambda} = e^{\lambda D^m} = e^{-\lambda F}$ where $F = -D^m$, which is not invertible, the generator of the sequence $P^{[\lambda]}_{n,m}(x)$ is $F = -D^m$. Now, if we consider another cross-sequence $M^{[\lambda]}_n(x)$ defined by (Rota 1973, §11)

$$M^{[\lambda]}_n(x) = (I - D)^{-\lambda} x^n = (-1)^n L^{(-\lambda-n)}_n(x),$$

then $G = [\log(I - D) + (-D^m)]$ will be the generator of the sequence

$$G^{[\lambda]}_{n,m}(x) = (I - D)^{-\lambda} P^{[\lambda]}_{n,m}(x) = e^{\lambda D^m} M^{[\lambda]}_n(x)$$

which itself is a cross-sequence with the delta operator D and relative to the invertible shift invariant operator $T_{m,\lambda} = (I - D)^{\lambda} \,.\, e^{-\lambda D^m}$. From the properties of the cross-sequence, we immediately infer that

$$G^{[\lambda+\mu]}_{n,m}(x) = (I - D)^{-\mu} G^{[\lambda]}_{n,m}(x) \qquad ...(4.1)$$

as well as

$$G^{[\lambda+\mu]}_{n,m}(x + y) = \sum_{k=0}^{n} \binom{n}{k} G^{[\lambda]}_{k,m}(x)\, G^{[\mu]}_{n-k,m}(y). \qquad ...(4.2)$$

Moreover, since $G^{[\lambda]}_{n,m}(x)$ is an Appell cross-sequence, the umbral composition formula for $G^{[\lambda]}_{n,m}(x)$ is

$$G^{[\lambda]}_{n,m}\,(G^{[\mu]}(x)) = G^{[\mu]}_{n,m}\,(G^{[\lambda]}(x)) = G^{[\lambda+\mu]}_{n,m}\,(x). \qquad \ldots(4.3)$$

The binomial theorem for Sheffer polynomials yields the identity

$$G^{(\lambda)}_{n,m}\,(x+y) = \sum_{k=0}^{\infty} \binom{n}{k} y^{n-k}\, G^{(\lambda)}_{k,m}\,(x). \qquad \ldots(4.4)$$

The generating function of $G^{(\lambda)}_{n,m}\,(x)$ is given by

$$\sum_{n \geqslant 0} \frac{G^{(\lambda)}_{n,m}\,(x)}{n!}\, t^n = \frac{1}{(1-t)^{\lambda}}\, e^{xt+\lambda t^m}. \qquad \ldots(4.5)$$

5. A Recurrence Relation for $P^{(\lambda)}_{n,m}\,(x)$

We use here the Pincherle derivative defined on the algebra of all shift invariant operators to obtain a recurrence relation for $P^{(\lambda)}_{n,m}\,(x)$.

We begin with

$$(e^{\lambda D^m})'\, f(x) = (e^{\lambda D^m}\, X - X e^{\lambda D^m})\, f(x)$$

where X is the multiplication operator. We get

$$\begin{aligned} e^{\lambda D^m}\, x\,.\, f(x) &= (e^{\lambda D^m})'\, f(x) + x\, e^{\lambda D^m}\, f(x) \\ &= m\, D^{m-1}\, \lambda\, e^{\lambda D^m}\, f(x) + x\, e^{\lambda D^m}\, f(x). \end{aligned}$$

If $f(x) = x^{n-1}$, then we have

$$\begin{aligned} e^{\lambda D^m}\, x^n &= m\lambda\, e^{\lambda D^m}\, D^{m-1}\, x^{n-1} + x\, e^{\lambda D^m}\, x^{n-1} \\ &= m\lambda (n-1)^{(m-1)}\, e^{\lambda D^m}\, x^{n-m} + x\, e^{\lambda D^m}\, x^{n-1} \end{aligned}$$

and hence,

$$P^{(\lambda)}_{n,m}\,(x) = \lambda\,.\,m! \binom{n-1}{m-1} e^{\lambda D^m}\, x^{n-m} + x\, P^{(\lambda)}_{n-1,m}\,(x).$$

So, we get a pure recurrence relation formula

$$P^{(\lambda)}_{n,m}\,(x) = \lambda\,.\,m! \binom{n-1}{m-1} P^{(\lambda)}_{n-m,m}\,(x) + x\, P^{(\lambda)}_{n-1,m}\,(x) \qquad \ldots(5.1)$$

which can be rewritten as

$$P^{(\lambda)}_{n,m}\,(x) = x\, P^{(\lambda)}_{n-1,m}\,(x) + \frac{\lambda m (n-1)!}{(n-m+1)!}\, [P^{(\lambda)}_{n-m,m}\,(x)]. \qquad \ldots(5.2)$$

Putting $\lambda = -\nu/2$ and $m = 2$, we get the following recurrence formula for $H_n^{(\nu)}(x)$:

$$H_n^{(\nu)}(x) = x\,H_{n-1}^{(\nu)}(x) - \nu\,H_{n-m+1}^{(\nu)}(x).$$

From (5.1) and (2.2) it follows that $P_{n,m}^{(\lambda)}(x)$ satisfies the mth order differential equation

$$m\lambda y_m + xy_1 - ny = 0 \qquad \text{...(5.3)}$$

which reduces for $m = 2$ and $\lambda = -\nu/2$ to a second order differential eqn. (5.4) for $H_n^{(\nu)}(x)$:

$$\nu y_2 - xy_1 + ny = 0 \ \text{ where } \ y = H_n^{(\nu)}(x). \qquad \text{...(5.4)}$$

Indeed, for $m = 2$, $\lambda = -\frac{1}{2}$, (5.3) reduces to the Hermite differential equation

$$y_2 - xy_1 + ny = 0. \qquad \text{...(5.5)}$$

Remark : Our notation for the ordinary Hermite polynomials $H_n(x)$ is the one used by Rota (1973) and corresponds to the notation in Rainville (1960) for

$$2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right).$$

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