LONGITUDINAL WAVE PROPAGATION IN A FINITE PIEZOELECTRIC CYLINDRICAL SHELL

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Axisymmetric vibrations of a hollow piezoelectric cylinder of finite length that belongs to (6 mm) crystal class or ceramics are investigated while the ends are constrained. Two cases are discussed by subjecting the exterior curved surface with time-dependent electrical boundary condition and mechanical boundary condition separately. The roots of the frequency equation for PZT-4 ceramic are calculated numerically.

INTRODUCTION

Applications of piezoelectric materials in various devices such as electric wave filters, fluid loaded transducers for sonar and ultrasonic cleaning, phonograph cartridges, force transducers, high voltage ignition, air-loaded transducers, displacement generators, piezoelectric transformers, piezoelectric pumps and surface wave filters are well known. Piezoelectric ceramics and polymer films in the form of a finite cylinder, plate or disc are used in many devices, because they possess high electromechanical coupling factors. In living bodies, bones, cartilage, tendons, ligaments and nerve tissues are piezoelectric substances. Some of them possess cylindrical shape. In an earlier analysis (Paul 1966), wave propagation in an infinite cylinder that belongs to (6 mm) class was investigated. In the present paper, we discuss axisymmetric vibrations in a finite hollow cylinder of the same piezoelectric material. Both the ends are constrained and the lateral surfaces of the cylinder are subjected to time-dependent electrical or mechanical boundary condition. Liu and Chang (1965) solved a similar kind of problem concerned with an isotropic elastic hollow cylinder by a method due to Mindlin and Goodman (1950). Recently, Paul and Sarma (1977) utilized the same technique to solve the torsional motion of piezoelectric (622) crystal class.

BASIC EQUATIONS

Since we consider axisymmetric vibrations of a piezoelectric cylindrical shell, we refer to the cylindrical polar coordinate system. $u(r, z, t)$ and $w(r, z, t)$ denote the components of displacement along r- and z-directions respectively. $\phi(r, z, t)$ represents the electric potential function. In the absence of the body force, equations of motion and the Gaussian equation for (6 mm) crystals or ceramics (∞, m) are (Paul 1967)

$$c_{11}(u_{,rr} + r^{-1} u_{,r} - r^{-2} u) + c_{44}u_{,zz} + (c_{44} + c_{13}) w_{,rz} + (e_{31} + e_{15}) \phi_{,rz} = su_{,tt} \quad ...(1)$$

$$(c_{44} + c_{13}) (u_{,rz} + r^{-1} u_{,z}) + c_{44}(w_{,rr} + r^{-1} w_{,r}) + c_{33}w_{,zz} + e_{33}\phi_{,zz} + e_{15}(\phi_{,rr} + r^{-1}\phi_{,r}) = sw_{,tt} \quad ...(2)$$

$$e_{15}(w_{,rr} + r^{-1}w_{,r}) + (e_{31} + e_{15}) (u_{,rz} + r^{-1}u_{,z}) + e_{33}w_{,zz} - \epsilon_{33}\phi_{,zz} - \epsilon_{11}(\phi_{,rr} + r^{-1}\phi_{,r}) = 0 \quad ...(3)$$

where c_{ij}, e_{ij} and ϵ_{ij} are the elastic, the piezoelectric and the dielectric constants respectively, and s is the density of the material. The comma followed by the subscripts denotes the partial derivatives with respect to those variables.

FORMULATION : CASE I

We assume that the cylinder is axially constrained at both ends represented by $z = 0$ and $z = L$, where L is the length of the cylinder. These ends are coated with electrodes that are shorted. Hence,

$$w(r, 0, t) = 0 = w(r, L, t);\ T_{rz}(r, 0, t) = 0 = T_{rz}(r, L, t);\ \phi(r, 0, t) = 0 = \phi(r, L, t). \quad ...(4)$$

The curved surfaces of the shell which are represented by $r = r_1$ and $r = r_2$ are kept traction-free :

$$T_{rr}(r_i, z, t) = 0 = T_{rz}(r_i, z, t);\ i = 1, 2. \quad ...(5)$$

We prescribe the electric potential on the exterior curved surface, while the interior curved surface is coated with the electrodes that are shorted. Hence,

$$\phi(r_2, z, t) = p(z, t);\ \phi(r_1, z, t) = 0. \quad ...(6)$$

For convenience we assume that

$$p(z, t) = \sum_{l=1}^{\infty} P_l(t) \sin (\alpha_l z) \quad ...(7)$$

where $\alpha_l = l\pi/L$.

Equations (1) – (7) formulate the problem of the first case. We proceed to obtain the solution by applying Mindlin and Goodman's technique with appropriate modifications.

DERIVATION OF SOLUTION

To obviate the time-dependency in the boundary condition (6), we proceed as was done by Liu and Chang (1965) and seek the solution of the system of differential equations (1) – (3) in the form

$$\left.\begin{aligned} u &= \sum_{l=1}^{\infty} [U_l(r, t) + G_l(r, t)] \cos (\alpha_l z) \\ w &= \sum_{l=1}^{\infty} [W_l(r, t) + H_l(r, t)] \sin (\alpha_l z) \\ \phi &= c_{44} e_{33}^{-1} \sum_{l=1}^{\infty} [E_l(r, t) + K_l(r, t)] \sin (\alpha_l z) \end{aligned}\right\} \quad ...(8)$$

where the functions G_l, H_l and K_l are to be determined in the sequel.

We may observe that the above form of the solution satisfies the end conditions (4) automatically. We shall use the following notations in the remaining analysis :

$$\bar{c}_{ij} = c_{ij}/c_{44},\ \bar{e}_{ij} = e_{ij}/e_{33},\ K_1^2 = e_{33}^2/(\epsilon_{11} c_{44}),\ K_3^2 = e_{33}^2/(\epsilon_{33} c_{44}),\ c_1^2 = c_{44}/s. \quad ...(9)$$

With the help of eqns. (8), the boundary conditions (5) and (6) transform to

$$\bar{c}_{11} U_{l,r} + \bar{c}_{12} r^{-1} U_l + \alpha_l(\bar{c}_{13} W_l + \bar{e}_{31} E_l) = -\bar{c}_{11} G_{l,r} - \bar{c}_{12} r^{-1} G_l - \alpha_l(\bar{c}_{13} H_l + \bar{e}_{31} K_l)$$

$$W_{l,r} - \alpha_l U_l + \bar{e}_{15} E_{l,r} = -H_{l,r} + \alpha_l G_l - \bar{e}_{15} K_{l,r};\ r = r_i,\ i = 1, 2.$$

$$E_l(r_1, t) = -K_l(r_1, t);\ E_l(r_2, t) = P_l(t) - K_l(r_2, t). \quad ...(10)$$

We demand that the functions G_l, H_l and K_l must be absent from eqns. (3) and (10). Hence, we get

$$\alpha_l(\bar{e}_{31} + \bar{e}_{15})(G_{l,r} + r^{-1} G_l) - \bar{e}_{15}(H_{l,rr} + r^{-1} H_{l,r}) + \alpha_l^2 (H_l - K_3^{-2} K_l) + K_1^{-2}(K_{l,rr} + r^{-1} K_{l,r}) = 0 \quad ...(11)$$

$$\bar{c}_{11} G_{l,r} + \bar{c}_{12} r^{-1} G_l + \alpha_l(\bar{c}_{13} H_l + \bar{e}_{31} K_l) = 0;$$

$$\alpha_l G_l - H_{l,r} - \bar{e}_{15} K_{l,r} = 0, \text{ for } r = r_1 \text{ and } r = r_2 \quad ...(12)$$

$$K_l(r_1, t) = 0,\ K_l(r_2, t) = P_l(t). \quad ...(13)$$

Since there is only a single differential eqn. (11) in the unknown functions G, H K, we need two more differential equations of similar type. In fact, the choice is unlimited, but we content ourselves with the following simple relations :

$$K_1^{-2}(K_{l,rr} + r^{-1} K_{l,r}) - K_3^{-2} \alpha_l^2 K_l = 0;\ G_{l,rr} + r^{-1} G_{l,r} - r^2 G_l = 0. \quad ...(14)$$

The motive behind this particular choice is that the former simplifies eqn. (11), while the latter eliminate a few terms of eqn. (1).

For simplicity, we put

$$G_l = g_l(r)\, P_l(t);\; H_l = [h_l(r) + v_l(r)]\, P_l(t);\; K_l = k_l(r)\, P_l(t). \qquad ...(15)$$

The decomposition of H_l is required to tackle the resulting non-homogeneous differential equation with non-homogeneous boundary condition. Substituting the results of the above equation in eqns. (11) – (14), we obtain, after rearrangement

$$K_1^{-2}(k_l'' + r^{-1}k_l') - \alpha_l^2\, K_3^{-2}\, k_l = 0,\; g_l'' + r^{-1}\, g_l' - r^{-2}g_l = 0, \qquad ...(16)$$

$$\alpha_l(\bar{e}_{31} + \bar{e}_{15})\, [g_l' + r^{-1}g_l] - \bar{e}_{15}(h_l'' + r^{-1}h_l') + \alpha_l^2\, h_l$$

$$= \bar{e}_{15}(v_l'' + r^{-1}v_l') - \alpha_l^2\, v_l \qquad ...(17)$$

$$k_l(r_1) = 0;\; k_l(r_2) = 1, \qquad ...(18)$$

$$\bar{c}_{11}\, [g_l']_{r=r_1} + \bar{c}_{12}r_1^{-1}g_l(r_1) + \alpha_l\bar{c}_{13}h_l(r_1) + \alpha_l\bar{c}_{13}v_l(r_1) = 0 \qquad ...(19)$$

$$\bar{c}_{11}\, [g_l']_{r=r_2} + \bar{c}_{12}r_2^{-1}\, g_l(r_2) + \alpha_l\bar{c}_{13}h_l(r_2) + \alpha_l\bar{c}_{13}v_l(r_2) = -\, \alpha_l\bar{e}_{31};$$

$$\alpha_l g_l(r_i) - [h_l']_{r=r_i} = \bar{e}_{15}[k_l']_{r=r_i} + [v_l']_{r=r_i}\, ;\; i = 1, 2 \qquad ...(20)$$

where the dash denotes the ordinary derivative with respect to r.

The solution of the first of eqns. (16) that satisfies the boundary conditions (18) is given by

$$k_l(r) = [K_0(ar_1)\, I_0(ar) - I_0(ar_1)\, K_0(ar)]/[K_0(ar_1)\, I_0(ar_2) - I_0(ar_1)\, K_0(ar_2)] \qquad ...(21)$$

where $a^2 = \alpha_l^2\, \epsilon_{33}/\epsilon_{11}$ and $I_0(r)$ and $K_0(r)$ are modified Bessel functions of order zero.

The general solution of the second of eqns. (16) is given by

$$g_l(r) = A_1 r + A_2 r^{-1} \qquad ...(22)$$

where A_1 and A_2 are arbitrary constants.

With the help of eqns. (22), the boundary conditions (19) may be simplified as

$$h_l(r_i) + v_l(r_i) = a_i;\; i = 1, 2 \qquad ...(23)$$

where

$$a_1 = [A_2 r_1^{-2}\, (\bar{c}_{11} - \bar{c}_{12}) - A_1(\bar{c}_{11} + \bar{c}_{12})]/\alpha_l\bar{c}_{13}, \qquad ...(24)$$

$$a_2 = ([A_2 r_2^{-2}\, (\bar{c}_{11} - \bar{c}_{12}) - A_1(\bar{c}_{11} + \bar{c}_{12})] - \alpha_l\bar{e}_{31})/(\alpha_l\bar{c}_{13}).$$

We choose $v_l(r)$ to be the solution of the homogeneous equation

$$(rv_l')' - \beta^2 r v_l = 0 \qquad ...(25)$$

with the boundary conditions

$$v_l(r_i) = a_i\,;\; i = 1, 2 \qquad ...(26)$$

where $\beta^2 = \alpha_l^2/\bar{e}_{15}$. Hence, we obtain

$$v_l(r) = [(a_1 K_0(\beta r_2) - a_2 K_0(\beta r_1))\, I_0(\beta r) - (a_1 I_0(\beta r_2) - a_2 I_0(\beta r_1))\, K_0(\beta r)]/ [I_0(\beta r_1)\, K_0(\beta r_2) - I_0(\beta r_2)\, K_0(\beta r_1)]. \qquad ...(27)$$

With the help of eqns. (22), (25) and (26), eqns. (17) and (23) reduce to the non-homogeneous equation

$$(rh_l')' - \beta^2 r h_l = 2A_1\alpha_l(1 + e_{31}/e_{15})r \qquad ...(28)$$

with homogeneous boundary conditions

$$h_l(r_i) = 0;\; i = 1, 2. \qquad ...(29)$$

Hence, we apply the Green's function technique, which is found in standard texts like Churchill (1958). Let $G(r, x)$ be the Green's function corresponding to the boundary value problem, determined by eqns. (28) and (29). We obtain

$$h_l(r) = 2\alpha_l A_1(1 + e_{31}/e_{15}) \int_{r_1}^{r_2} xG(r, x)\, dx. \qquad ...(30)$$

The unknown constants A_1 and A_2 are now fixed, with the help of eqn. (20).

Equations (15) – (30) enable us to specify unambiguously the functions G, H and K. The desired choice of the functions G, H and K reduces eqns. (1) – (3) to the following :

$$\bar{c}_{11}(U_{l,rr} + r^{-1}U_{l,r} - r^{-2}U_l) + \alpha_l[-\alpha_l U_l + (1 + \bar{c}_{13})\, W_{l,r} + (\bar{e}_{31} + \bar{e}_{15})\, E_{l,r}] - c_1^{-2}\, U_{l,tt} = c_1^{-2}\, G_{l,tt} + \alpha_l[\alpha_l G_l - (1 + \bar{c}_{13}) H_{l,r} - (\bar{e}_{31} + \bar{e}_{15})\, K_{l,r}] \qquad ...(31)$$

$$-\alpha_l(1 + \bar{c}_{13})(U_{l,r} + r^{-1}U_l) + W_{l,rr} + r^{-1}W_{l,r} - \alpha_l^2(\bar{c}_{33}\, W_l + E_l) + \bar{e}_{15}(E_{l,rr} + r^{-1}E_{l,r}) - c_1^{-2}\, W_{l,tt} = c_1^{-2}\, H_{l,tt} + \alpha_l(1 + \bar{c}_{13})(G_{l,r} + r^{-1}G_l) - H_{l,rr} - r^{-1}H_{l,r} + \alpha_l^2(\bar{c}_{33}H_l + K_l) - \bar{e}_{15}(K_{l,rr} + r^{-1}K_{l,r}) \qquad ...(32)$$

$$\bar{e}_{15}(W_{l,rr} + r^{-1}W_{l,r}) - \alpha_l(\bar{e}_{31} + \bar{e}_{15})(U_{l,r} + r^{-1}U_l) - \alpha_l^2 (W_l - K_3^{-2}E_l)$$
$$- K_1^{-2}(E_{l,rr} + r^{-1}E_{l,r}) = 0 \qquad \ldots(33)$$

and renders the boundary conditions (10) homogeneous, as follows :

$$\bar{c}_{11}U_{l,r} + \bar{c}_{12}r^{-1}U_l + \alpha_l \bar{c}_{13}W_l = 0,$$
$$W_{l,r} - \alpha_l U_l + \bar{e}_{15}E_{l,r} = 0 = E_l \text{ ; for } r = r_i \text{ ; } i = 1, 2. \qquad \ldots(34)$$

Characteristic Functions

We proceed to solve the completely homogeneous system of equations obtained by dropping the terms in the right hand side of eqns. (31) and (32). To this end, we make the transformations

$$U_l = \sum_{n=1}^{\infty} \exp(iw_{ln}t)\, U_{ln}(r); \; W_l = \sum_{n=1}^{\infty} \exp(iw_{ln}t)\, W_{ln}(r);$$
$$E_l = \sum_{n=1}^{\infty} \exp(iw_{ln}t)\, E_{ln}(r) \qquad \ldots(35)$$

into the homogeneous system and have

$$\left.\begin{aligned}
&\bar{c}_{11}(U''_{ln} + r^{-1}U'_{ln} - r^{-2}U_{ln}) + \alpha_l(-\alpha_l U_{ln} + (1 + \bar{c}_{13})\, W'_{ln} \\
&\quad + (\bar{e}_{31} + \bar{e}_{15})\, E'_{ln}) + c_1^{-2}\, w_{ln}^2 U_{ln} = 0 \\
&- \alpha_l(1 + \bar{c}_{13})(U'_{ln} + r^{-1}U_{ln}) + W'_{ln} + r^{-1}W'_{ln} - \alpha_l^2 (\bar{c}_{33}W_{ln} + E_{ln}) \\
&\quad + \bar{e}_{15}(E''_{ln} + r^{-1}E'_{ln}) + c_1^{-2} w_{ln}^2 W_{ln} = 0 \\
&\bar{e}_{15}(W''_{ln} + r^{-1}W'_{ln}) - \alpha_l(\bar{e}_{31} + \bar{e}_{15})(U'_{ln} + r^{-1}U_{ln}) \\
&\quad - \alpha_l^2 (W_{ln} - K_3^{-2} E_{ln}) - K_1^{-2}(E''_{ln} + r^{-1} E'_{ln}) = 0.
\end{aligned}\right\} \ldots(36)$$

The boundary conditions (34) are transformed as

$$\bar{c}_{11}U'_{ln} + \bar{c}_{12}r^{-1}U_{ln} + \bar{c}_{13}\alpha_l W_{ln} = 0;$$
$$W'_{ln} - \alpha_l U_{ln} + \bar{e}_{15}E'_{ln} = 0 = E_{ln} \text{ for } r = r_i;\; i = 1, 2. \qquad \ldots(37)$$

Let β_i^2 ; $i = 1, 2, 3$ be the roots of the following determinantal equation which is cubic in β^2

$$\begin{vmatrix} (c_1^{-2} w_{ln}^2 - \alpha_l^2) - \bar{c}_{11}\beta^2 & \alpha_l(1 + \bar{c}_{13}) & \alpha_l(\bar{e}_{31} + \bar{e}_{15}) \\ \alpha_l(1 + \bar{c}_{13})\,\beta^2 & (c_1^{-2} w_{ln}^2 - \alpha_l^2 \bar{c}_{33}) - \beta^2 & -(\bar{e}_{15}\beta^2 + \alpha_l^2) \\ \alpha_l(\bar{e}_{31} + \bar{e}_{15})\,\beta^2 & -(\bar{e}_{15}\beta^2 + \alpha_l^2) & (K_1^{-2}\beta^2 + K_3^{-2}\alpha_l^2) \end{vmatrix} = 0. \quad \text{...(38)}$$

Then the solution of the system of simultaneous differential eqns. (36) can be shown as

$$\left.\begin{aligned} U_{ln} &= \sum_{i=1}^{3} [A_{lni}\, J_1(\beta_i r) + B_{lni}\, Y_1(\beta_i r)] \\ W_{ln} &= \sum_{i=1}^{3} d_{lni}\, [A_{lni}\, J_0(\beta_i r) + B_{lni}\, Y_0(\beta_i r)] \\ E_{ln} &= \sum_{i=1}^{3} e_{lni}\, [A_{lni}\, J_0(\beta_i r) + B_{lni}\, Y_0(\beta_i r)] \end{aligned}\right\} \quad \text{...(39)}$$

where $J_n(r)$ and $Y_n(r)$ are Bessel functions of order n.

A_{lni} and B_{lni} are arbitrary constants and the constants d_{lni} and e_{lni} are given by

$$\alpha_l\beta_i\,[(1 + \bar{c}_{13})\,d_{lni} + (\bar{e}_{31} + \bar{e}_{15})\,e_{lni}] = c_1^{-2}\, w_{ln}^2 - \bar{c}_{11}\beta_i^2 - \alpha_l^2;$$

$$(c_1^{-2}\, w_{ln}^2 - \alpha_l^2\, \bar{c}_{33} - \beta_i^2)\,d_{lni} - (\bar{e}_{15}\beta_i^2 + \alpha_l^2)\,e_{lni} = (1 + \bar{c}_{13})\,\alpha_l\beta_i;$$

$$\text{for } i = 1, 2, 3. \quad \text{...(40)}$$

We substitute the results of eqns. (39) in the boundary conditions (37) and eliminate the unknown constants A_{lni} and B_{lni}, to obtain the frequency equation

$$|\, a_{ij} \,| = 0, \quad i, j = 1, 2, \ldots, 6. \quad \text{...(41)}$$

The elements of the first three rows that occur in the first three columns of the above determinant are given by

$$a_{1j} = (\bar{c}_{11} + \bar{c}_{12})\, r_1^{-1}\, J_1(\beta_j r_1) - \bar{c}_{11}\beta_j J_2(r_1\beta_j) + \bar{c}_{13}\alpha_l d_{lnj}\, J_0(\beta_j r_1);$$

$$a_{2j} = [\alpha_l + \beta_j(d_{lnj} + \bar{e}_{15}e_{lnj})]\, J_1(\beta_j r_1); \quad a_{3j} = e_{lnj}\, J_0(\beta_j r_1); \quad j = 1, 2, 3. \quad \text{...(42)}$$

The elements in the remaining columns of the first three rows are obtained from those of the first three columns by writing the Bessel functions of the second kind, in place of those of the first kind. The elements of the remaining rows are simply obtained from the first three rows, in the same order on just replacing r_1 by r_2, while the other quantities are unaltered.

NUMERICAL RESULTS

For numerical calculations, we consider a hollow cylinder of piezoelectric ceramic PZT-4 having the size $r_2/r_1 = 6$, $r_1/L = 0.1$.

The following physical constants are taken from Mason (1964):

$c_{11} = 13.9 \times 10^{10}$ N/m² $\quad e_{15} = 12.7$ C/m²

$c_{12} = 7.78 \times$,, $\quad e_{31} = -5.2$,,

$c_{13} = 7.43 \times$,, $\quad e_{33} = 15.1$,,

$c_{33} = 11.5 \times$,, $\quad \epsilon_{11} = 730 \times 10^{-12}$ F/m

$c_{44} = 2.56 \times$,, $\quad \epsilon_{33} = 635 \times$,,

The programme to evaluate the roots w_{ln} of the frequency equation (41) is made on the digital computer IBM 370/155. The first few values of Lw_{ln}/c_1 are listed in Table I.

We have adopted the following iterative procedure for numerical computations. For a fixed value of l, we evaluate the determinant, present in the left hand side of eqn. (41), for various values of the unknown quantity Lw_{ln}/c_1, commencing with the initial value zero and each time adding a fixed but small increment to that unknown quantity till the value of the determinant changes its sign. Then the bisection method is applied to locate the root correct to a chosen number of decimal places. With this root as the initial value, the procedure is repeated to find the next root.

When two roots of eqn. (38), cubic in β^2, are complex conjugate, the above determinant is replaced by an appropriate one. When a root of the cubic equation is negative, the Bessel functions involving this root are replaced by the corresponding modified Bessel functions with suitable changes.

TABLE I

Values of Lw_{ln}/c_1

n \ l	1	2	3	4
1	3.1416	6.8036	7.3356	14.9371
2	4.5152	14.4741	11.8292	27.2233
3	6.9620	16.9384	13.0266	28.2471
4	10.2879	20.8773	14.0428	32.5661
5	13.4261	21.5061	23.1226	35.4288

The Complete Solution

We observe that the solution of the completely homogeneous systems, known as characteristic functions, satisfies the biorthogonal condition, viz.,

$$\int_{r_1}^{r_2} r(U_{lm}U_{ln} + W_{lm}W_{ln})\, dr = 0, \text{ if } m \neq n. \quad \ldots(43)$$

We put $$\theta_{ln} = \int_{r_1}^{r_2} r(U_{ln}^2 + W_{ln}^2)\, dr \neq 0. \quad \ldots(44)$$

We establish the biorthogonal property in the Appendix.

We proceed to solve the system of non-homogeneous eqns. (31) – (33) by means of the characteristic functions found in eqns. (39).

We apply the transformations

$$U_l = \sum_{n=1}^{\infty} T_{ln}(t)\, U_{ln}(r); \quad W_l = \sum_{n=1}^{\infty} T_{ln}(t)\, W_{ln}(r);$$

$$E_l = \sum_{n=1}^{\infty} T_{ln}(t)\, E_{ln}(r) \quad \ldots(45)$$

into the system of eqns. (31) and (32) and obtain

$$-\, c_1^{-2} \sum_{n=1}^{\infty} (\ddot{T}_{ln} + w_{ln}^2\, T_{ln})\, U_{ln} = c_1^{-2}\, g_l \ddot{P}_l + P_l f_{l1}(r) \quad \ldots(46)$$

$$-\, c_1^{-2} \sum_{n=1}^{\infty} (\ddot{T}_{ln} + w_{ln}^2\, T_{ln})\, W_{ln} = c_1^{-2}\, \bar{h}_l \ddot{P}_l + P_l f_{l2}(r) \quad \ldots(47)$$

where the dot denotes the ordinary derivative with respect to time t and

$$\left.\begin{aligned} f_{l1}(r) &= [\alpha_l g_l - (1 + \bar{c}_{13})(h_l' + v_l') - (\bar{e}_{31} + \bar{e}_{15})\, k_l']\, \alpha_l \\ f_{l2}(r) &= \alpha_l(1 + \bar{c}_{13})(g_l' + r^{-1}g_l) - (h_l'' + r^{-1}h_l') \\ &\quad + (\alpha_l^2\, \bar{c}_{33} - \beta^2)\, v_l + \alpha_l^2\, (\bar{c}_{33}h_l + k_l) - \bar{e}_{15}(k_l'' + r^{-1}k_l'). \end{aligned}\right\} \quad \ldots(48)$$

$$\bar{h}_l = h_l + v_l$$

Equation (33) identically vanishes.

The biorthogonal property of the characteristic functions expressed in eqns. (43) and (44) enables us to obtain generalized Fourier series type expansion for the functions that occur in the right hand side of eqns. (46) and (47), as follows :

$$\left.\begin{aligned} g_l(r) &= \sum_{n=1}^{\infty} b_{ln} U_{ln}(r); & \bar{h}_l(r) &= \sum_{n=1}^{\infty} b_{ln} W_{ln}(r); \\ f_{l1}(r) &= c_1^{-2} \sum_{n=1}^{\infty} c_{ln} U_{ln}(r); & f_{l2}(r) &= c_1^{-2} \sum_{n=1}^{\infty} c_{ln} W_{ln}(r). \end{aligned}\right\} \quad \ldots(49)$$

With the help of eqns. (43) and (44), the coefficients may be obtained as

$$b_{ln} = \theta_{ln}^{-1} \int_{r_1}^{r_2} (g_l U_{ln} + \bar{h}_l W_{ln})\, dr, \quad c_{ln} = \theta_{ln}^{-1} c_1^2 \int_{r_1}^{r_2} r(f_{l1} U_{ln} + f_{l2} W_{ln})\, dr. \quad \ldots(50)$$

We substitute the results of eqns. (49) into (46) and (47) and equate the coefficients of the nth terms on both sides :

$$\ddot{T}_{ln} + w_{ln}^2 T_{ln} = q_{ln}, \text{ where } q_{ln} = -(b_{ln}\ddot{P}_l + \alpha_l^2 P_l c_{ln}). \quad \ldots(51)$$

Solution of eqn. (51) is

$$T_{ln} = M_{ln} \cos(w_{ln}t) + N_{ln} \sin(w_{ln}t) + w_{ln}^{-1} \int_0^t q_{ln}(x) \sin[w_{ln}(t-x)]\, dx \quad \ldots(52)$$

where M_{ln} and N_{ln} are arbitrary constants to be fixed by the initial conditions.

With the help of eqns. (8), (15), (45), (49) and (52), the required solution is obtained as

$$u = \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} [M_{ln} \cos(w_{ln}t) + N_{ln} \sin(w_{ln}t) + w_{ln}^{-1} \int_0^t q_{ln}(x) \sin(w_{ln}(t-x))\, dx + b_{ln} P_l]\, U_{ln} \cos(\alpha_l z) \quad \ldots(53)$$

$$w = \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} [M_{ln} \cos(w_{ln}t) + N_{ln} \sin(w_{ln}t) + w_{ln}^{-1} \int_0^t q_{ln}(x) \sin(w_{ln}(t-x))\, dx + b_{ln} P_l]\, W_{ln} \sin(\alpha_l z), \quad \ldots(54)$$

$$\phi = c_{44} e_{33}^{-1} \sum_{l=1}^{\infty} \Big(\sum_{n=1}^{\infty} [M_{ln} \cos(w_{ln}t) + N_{ln} \sin(w_{ln}t) + w_{ln}^{-1} \int_0^t q_{ln}(x) \sin(w_{ln}(t-x))\, dx\,]\, E_{ln} + P_l k_l \Big) \sin(\alpha_l z). \quad \ldots(55)$$

The Initial Conditions

The initial conditions, in general, may be stated as

$$u(r, z, 0) = F_1(r, z); \qquad w(r, z, 0) = f_1(r, z)$$

$$[\dot{u}(r, z, t)]_{t=0} = F_2(r, z); \qquad [\dot{w}(r, z, t)]_{t=0} = f_2(r, z). \qquad \ldots(56)$$

We assume that the functions F_i, f_i, $i = 1, 2$ possess the following infinite series expansion :

$$F_i = \sum_{l=1}^{\infty} R_{li}(r) \cos(\alpha_l z); \quad f_i = \sum_{l=1}^{\infty} S_{li}(r) \sin(\alpha_l z); \quad i = 1, 2 \qquad \ldots(57)$$

whose coefficients are given by

$$R_{li} = 2L^{-1} \int_0^L F_i(r, z) \cos(\alpha_l z)\, dz;$$

$$S_{li} = 2L^{-1} \int_0^L f_i(r, z) \sin(\alpha_l z)\, dz; \quad i = 1, 2. \qquad \ldots(58)$$

With the help of eqns. (53) - (57), we obtain

$$R_{l_1} = \sum_{n=1}^{\infty} (b_{ln} P_l(0) + M_{ln})\, U_{ln};$$

$$R_{l_2} = \sum_{n=1}^{\infty} (b_{ln} \dot{P}_l(0) + w_{ln} N_{ln})\, U_{ln},$$

$$S_{l_1} = \sum_{n=1}^{\infty} (b_{ln} P_l(0) + M_{ln})\, W_{ln};$$

$$S_{l_2} = \sum_{n=1}^{\infty} (b_{ln} \dot{P}_l(0) + w_{ln} N_{ln})\, W_{ln} \qquad \ldots(59)$$

where we have put $\dot{P}_l(0) = [\dot{P}_l(t)]_{t=0}$.

Finally, we can obtain the constants of integration, M_{ln} and N_{ln}, by applying the biorthogonal conditions (43) and (44).

$$\left.\begin{aligned} b_{ln} P_l(0) + M_{ln} &= \theta_{ln}^{-1} \int_{r_1}^{r_2} r(R_{l_1} U_{ln} + S_{l_1} W_{ln})\, dr \\ b_{ln} \dot{P}_l(0) + w_{ln} N_{ln} &= \theta_{ln}^{-1} \int_{r_1}^{r_2} r(R_{l_2} U_{ln} + S_{l_2} W_{ln})\, dr. \end{aligned}\right\} \qquad \ldots(60)$$

Hence, the constants M_{ln} and N_{ln} present in eqns. (53) – (55) are determined. This enables us to know the stress field and the electric field in the cylindrical shell.

Formulation and Solution of the Problem—Case II

We establish in the present section that the electric potential function need not be decomposed into two parts, when we prescribe time-dependent normal traction on any one of the curved surfaces of the piezoelectric cylindrical shell executing axisymmetric vibrations. To be definite, we prescribe the normal stress on the exterior curved surface. Hence,

$$T_{rr}(r_2, z, t) = c_{44}\, p(z, t); \;\; T_{rr}(r_1, z, t) = 0;$$
$$T_{rz}(r_i, z, t) = 0; \;\; i = 1, 2. \qquad \ldots(61)$$

We consider the simple electrical boundary condition wherein both the curved surfaces are coated with electrodes that are shorted. Hence,

$$\phi(r_i, z, t) = 0; \quad i = 1, 2. \qquad \ldots(62)$$

We consider the same end conditions as expressed in eqns. (4).

We assume that the external force function $p(z, t)$ has the following expansion :

$$p(z, t) = \sum_{l=1}^{\infty} P_l(t) \cos(\alpha_l z). \qquad \ldots(63)$$

Equations (1) - (4) and (61) - (63) determine the problems of the second case. We seek a solution of the form

$$\left.\begin{aligned} u &= \sum_{l=1}^{\infty} [U_l(r, t) + G_l(r, t)] \cos(\alpha_l z) \\ w &= \sum_{l=1}^{\infty} [W_l(r, t) + H_l(r, t)] \sin(\alpha_l z) \\ \phi &= c_{44} e_{33}^{-1} \sum_{l=1}^{\infty} E_l(r, t) \sin(\alpha_l z). \end{aligned}\right\} \qquad \ldots(64)$$

As before, we choose the functions G_l and H_l such that they will be absent from eqn. (3) and boundary conditions (61). Hence, we get

$$\bar{e}_{15}(H_{l,rr} + r^{-1}H_{l,r}) - \alpha_l^2 H_l - \alpha_l(\bar{e}_{31} + \bar{e}_{15})(G_{l,r} + r^{-1}G_l) = 0 \qquad \ldots(65)$$

$$\begin{aligned} \bar{c}_{11}(G_{l,r}) + \bar{c}_{12} r^{-1} G_l + \alpha_l \bar{c}_{13} H_l &= P_l(t) \;\text{ if }\; r = r_2 \\ &= 0 \qquad \text{if }\; r = r_1 \end{aligned} \qquad \ldots(66)$$

$$\alpha_l G_l - H_{l,r} = 0 \;\text{ for }\; r = r_i, \; i = 1, 2.$$

As there is only a single partial differential eqn. (65), involving two unknown functions, we consider the following relation

$$G_{l,rr} + r^{-1}G_{l,r} - r^{-2}G_l = 0. \qquad \ldots(67)$$

The choice is not only a simple one but also simplifies eqn. (1). For convenience, we put

$$G_l(r, t) = g_l(r)\, P_l(t); \quad H_l(r, t) = [h_l(r) + v_l(r)]\, P_l(t). \qquad \ldots(68)$$

The rest of the procedure is the same as in the previous case. However, in the present case, the function $K_l(r, t)$ does not appear in the analysis. We observe that the second of eqns. (24) will be modified as

$$a_2 = ([A_2 r_2^{-2} (\bar{c}_{11} - \bar{c}_{12}) - A_1(\bar{c}_{11} + \bar{c}_{12})] + 1)/\alpha_l \bar{c}_{13}. \qquad \ldots(69)$$

The arbitrary constants A_1 and A_2 are determined from the equation

$$\alpha_l g_l - h_l' = v_l' \text{ for } r = r_1 \text{ and } r = r_2.$$

In place of eqn. (48), we find

$$\left.\begin{aligned} f_{l1} &= (\alpha_l g_l - (1 + \bar{c}_{13})(h_l' + v_l'))\, \alpha_l \\ f_{l2} &= \alpha_l(1 + \bar{c}_{13})(g_l' + r^{-1} g_l) - (h_l'' + r^{-1} h_l') + \alpha_l^2 \bar{c}_{33} h_l \\ &\quad + \alpha_l^2 \bar{c}_{33} - \beta^2)\, v_l. \end{aligned}\right\} \qquad \ldots(70)$$

Hence, the problem of the present case is completely solved. We obtain an identical expression for the components u and w of the displacement present in eqns. (53) and (54). We derive the electric potential function as

$$\phi = c_{44} e_{33}^{-1} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} [M_{ln} \cos(w_{ln} t) + N_{ln} \sin(w_{ln} t)$$

$$+ w_{ln}^{-1} \int_0^t q_{ln}(x) \sin(w_{ln}(t - x))\, dx]\, E_{ln} \sin(\alpha_l z). \qquad \ldots(71)$$

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APPENDIX

We proceed to establish the biorthogonal property of the characteristic functions, stated in eqns. (43) and (44).

We multiply the first one of eqns. (36) throughout with U_{lm} and rearrange as follows :

$$(\alpha_l^2 - c_1^{-2} w_{ln}^2) U_{ln}U_{lm} = \bar{c}_{11}(U''_{ln} U_{lm} + r^{-1}U'_{ln} U_{lm} - r^{-2}U_{ln}U_{lm})$$

$$+ \alpha_l [(1 + \bar{c}_{13}) W'_{ln} U_{lm} + (\bar{e}_{31} + \bar{e}_{15}) E'_{ln} U_{lm}] \quad \ldots(\text{A1})$$

where we assume that $m \neq n$.

From the above equation, we subtract a similar equation obtained by interchanging the suffixes m and n of eqn. (A1), to get

$$c_1^{-2} (w_{lm}^2 - w_{ln}^2) U_{lm}U_{ln} = \bar{c}_{11} [(U''_{ln} U_{lm} - U''_{lm} U_{ln})$$

$$+ r^{-1}(U'_{ln} U_{lm} - U'_{lm}U_{ln})]$$

$$+ \alpha_l [(1 + \bar{c}_{13}) (W'_{ln} U_{lm} - W'_{lm} U_{ln}) + (\bar{e}_{31} + \bar{e}_{15})(E'_{ln} U_{lm} - E'_{lm}U_{ln})]. \quad \ldots(\text{A2})$$

We repeat the same procedure for the second one of eqns. (36) and obtain

$$c_1^{-2} (w_{lm}^2 - w_{ln}^2) W_{lm}W_{ln} = \alpha_l(1 + \bar{c}_{13}) [(U'_{lm} W_{ln} - U'_{ln} W_{lm})$$

$$+ r^{-1}(U_{lm}W_{ln} - U_{ln}W_{lm})] + (W''_{ln} W_{lm} - W''_{lm} W_{ln})$$

$$+ r^{-1}(W'_{ln} W_{lm} - W'_{lm} W_{ln}) + \alpha_1^2 (E_{lm}W_{ln} - E_{ln}W_{lm})$$

$$+ \bar{e}_{15} [(E''_{ln} W_{lm} - E''_{lm} W_{ln}) + r^{-1}(E'_{ln} W_{lm} - E'_{lm} W_{ln})]. \quad \ldots(\text{A3})$$

In the same way, we can obtain from the last one of eqns. (36)

$$\bar{e}_{15} [(W''_{ln} E_{lm} - W''_{lm} E_{ln}) + r^{-1}(W'_{ln} E_{lm} - W'_{lm} E_{ln})]$$

$$+ \alpha_l(\bar{e}_{31} + \bar{e}_{15}) [(U'_{lm} E_{ln} - U'_{ln} E_{lm}) + r^{-1}(U_{lm}E_{ln} - U_{ln}E_{lm})]$$

$$+ \alpha_l^2 (W_{lm}E_{ln} - W_{ln}E_{lm}) + K_1^{-2} [(E''_{lm} E_{ln} - E''_{ln} E_{lm})$$

$$+ r^{-1}(E'_{lm} E_{ln} - E'_{ln} E_{lm})] = 0. \quad \ldots(\text{A4})$$

We multiply eqns. (A2) - (A4) with r, add them and finally integrate with respect to r in the interval (r_1, r_2), to obtain

$$c_1^{-2}(w_{lm}^2 - w_{ln}^2)\int_{r_1}^{r_2} r(U_{ln}U_{lm} + W_{lm}W_{ln})\,dr$$

$$= \int_{r_1}^{r_2} \{\bar{c}_{11}[r(U''_{ln}U_{lm} - U''_{lm}U_{ln}) + (U'_{ln}U_{lm} - U'_{lm}U_{ln})]$$

$$+ \alpha_l r[(1+\bar{c}_{13})(W'_{ln}U_{lm} - W'_{lm}U_{ln}) + (\bar{e}_{31}+\bar{e}_{15})(E'_{ln}U_{lm} - E'_{lm}U_{ln})]$$

$$+ \alpha_l(1+\bar{c}_{13})[r(U'_{lm}W_{ln} - U'_{ln}W_{lm})$$

$$+ (U_{lm}W_{ln} - U_{ln}W_{lm})] + r(W''_{ln}W_{lm} - W''_{lm}W_{ln})$$

$$+ (W'_{ln}W_{lm} - W'_{lm}W_{ln}) + \bar{e}_{15}[r(E''_{ln}W_{lm} - E''_{lm}W_{ln})$$

$$+ (E'_{ln}W_{lm} - E'_{lm}W_{ln})] + \bar{e}_{15}[r(W''_{ln}E_{lm} - W''_{lm}E_{ln})$$

$$+ (W'_{ln}E_{lm} - W'_{lm}E_{ln})] + \alpha_l(\bar{e}_{31}+\bar{e}_{15})[r(U'_{lm}E_{ln} - U'_{ln}E_{lm})$$

$$+ (U_{lm}E_{ln} - U_{ln}E_{lm})] + K_1^{-2}[r(E'_{lm}E_{ln} - E'_{ln}E_{lm})]\}\,dr. \qquad \text{...(A5)}$$

The right-hand side of the above equation integrates out easily and turns out to be

$$[r\{\bar{c}_{11}(U'_{ln}U_{lm} - U'_{lm}U_{ln}) + \alpha_l(1+\bar{c}_{13})(U_{lm}W_{ln} - U_{ln}W_{lm})$$

$$+ \alpha_l(\bar{e}_{31}+\bar{e}_{15})(U_{lm}E_{ln} - U_{ln}E_{lm})$$

$$+ (W'_{ln}W_{lm} - W'_{lm}W_{ln}) + \bar{e}_{15}(E'_{ln}W_{lm} - E'_{lm}W_{ln}$$

$$+ W'_{ln}E_{lm} - W'_{lm}E_{ln}) + K_1^{-2}(E'_{lm}E_{ln} - E'_{ln}E_{lm})\}]_{r_1}^{r_2}. \qquad \text{...(A6)}$$

The expression (A6) vanishes with the help of the boundary conditions (37).

Since $w_{lm} \neq w_{ln}$, from the result of eqns. (A5), we obtain

$$\int_{r_1}^{r_2} r(U_{lm}U_{ln} + W_{lm}W_{ln})\,dr = 0; \quad \text{if } m \neq n.$$