FIXED POINT THEOREMS IN BANACH SPACES

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Some fixed point theorems in Banach spaces for a new type of contractive mapping have been presented.

§1. Let X denote a Banach space. A mapping $T : X \to X$ is said to be nonexpansive if $\| Tx - Ty \| \leqslant \| x - y \|$ for all $x, y \in X$. Many fixed point theorems for nonexpansive mappings have been derived in recent years. (For related results *see* Browder and Petryshyn 1966, Belluse and Kirk 1966, Diaz and Metcalf 1969, Kirk 1965, 1971, Petryshyn 1971 and Petryshyn and Williamson 1973). The object of this paper is to prove some fixed point theorems using a symmetric rational fraction. We shall be concerned with a mapping, which satisfies the following contractive condition.

Let T be a mapping of X into itself, such that

$$\| Tx - Ty \| \leqslant \left[\frac{\| x - Tx \| \, \| x - Ty \| + \| y - Ty \| \, \| y - Tx \|}{\| x - Ty \| + \| y - Tx \|} \right] \qquad \ldots(1)$$

for all $x, y \in X$, $x \neq y$.

Definitions — For a bounded set $K \subset X$, the diameter of K denoted by $D(K)$ is defined as

$$D(K) = \sup \{\| x - y \| : x, y \in K).$$

A bounded convex subset K of a Banach space is said to have a normal structure, if for each convex subset H of K with more than one point, there is a point $x \in H$ such that

$$\sup \{ \| x - y \| : y \in H \} < D(H).$$

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The convex hull of K is denoted by $co(K)$ and the closed convex hull of K is denoted by $\overline{co}(K)$.

Let

$$O(x) = \{x, Tx, T^2x, \dots\} \text{ where } T : K \to K$$

and

$$T(K) = \{Tx : x \in K\}.$$

§2. We first prove the following lemma.

Lemma — Let K be a subset of a Banach space X and let T be a mapping of K into itself, such that for $x \in K$, T satisfies (1). Then

$$\| T^n x - T^{n+1} x \| \leqslant \| T^{n-1} x - T^n x \|$$

and for positive integers m and n,

$$\| T^n x - T^m x \| \leqslant \| x - Tx \|.$$

PROOF :

$\| T^n x - T^{n+1} x \|$

$$\leqslant \left[\frac{\| T^{n-1}x - T^n x \| \, \| T^{n-1}x - T^{n+1}x \| + \| T^n x - T^{n+1} x \| \, \| T^n x - T^n x \|}{\| T^{n-1}x - T^{n+1}x \| + \| T^n x - T^n x \|} \right].$$

Thus,

$$\| T^n x - T^{n+1} x \| \leqslant \| T^{n-1} x - T^n x \| \leqslant \dots \leqslant \| x - Tx \|.$$

Now

$\| T^n x - T^m x \|$

$$\leqslant \left[\frac{\| T^{n-1}x - T^n x \| \, \| T^{n-1}x - T^m x \| + \| T^{m-1}x - T^m x \| \, \| T^{m-1}x - T^n x \|}{\| T^{n-1}x - T^m x \| + \| T^{m-1}x - T^n x \|} \right]$$

$$\leqslant \left[\frac{\| x - Tx \| \, \| T^{n-1}x - T^m x \| + \| x - Tx \| \, \| T^{m-1}x - T^n x \|}{\| T^{n-1}x - T^m x \| + \| T^{m-1}x - T^n x \|} \right]$$

i.e. $\| T^n x - T^m x \| \leqslant \| x - Tx \|$.

Theorem 1 — Let K be a nonempty, bounded, closed, convex subset of a reflexive Banach space X and let K have normal structure. If $T : K \to K$ is continuous and satisfies (1), then T has a unique fixed point in K.

PROOF : Since X is a reflexive Banach space, every descending chain of nonempty closed convex subsets of X has nonempty intersection (Kirk 1965). Hence, we may use Zorn's lemma to obtain a subset K_1 of K minimal with respect to being closed, convex and invariant under T.

If $D(K_1) = 0$, we are home. Suppose $D(K_1) > 0$.

Since K has a normal structure, there is a point $y \in K_1$, such that

$$\sup \{ \| x - y \| ; x \in K_1 \} \leqslant r < D(K_1).$$

Thus, $\| y - Ty \| \leqslant r$ and by the above lemma, $D(O(y)) \leqslant r$. Let $H = \{x \in K_1 : D(O(x)) \leqslant r\}$ and let $G = \overline{co}(T(H))$. Then G is closed, convex and nonempty. Let $g \in G$. Then there are three cases to consider.

Case 1 — $g = Th$ for some $h \in H$.

Then

$$\| g - Tg \| = \| Th - T^2 h \| \leqslant \| h - Th \| \leqslant r.$$

Hence,

$$g \in H \text{ and } Tg \in G.$$

Case 2 — Let

$$g = \sum_{i=1}^{n} \lambda_i Th_i, \; h_i \in H, \; \lambda_i \geqslant 0, \; \sum_{i=1}^{n} \lambda_i = 1.$$

Then

$$\| Tg - g \| = \| Tg - \sum_{i=1}^{n} \lambda_i Th_i \|$$

$$\leqslant \sum_{i=1}^{n} \lambda_i \| T^2 h_i - Th_i \| \leqslant r.$$

So

$$\| g - Tg \| \leqslant r, \; g \in H \text{ and } Tg \in G.$$

Case 3 — g is the limit of terms of the form

$$\sum_{i=1}^{n} \lambda_i Th_i$$

where

$$\lambda_i \geqslant 0, \; h_i \in H, \; \sum_{i=1}^{n} \lambda_i = 1.$$

Then for any such term we have

$$\| Tg - g \| \leqslant \| Tg - \sum_{i=1}^{n} T^2 \lambda_i h_i \| + \| \sum_{i=1}^{n} \lambda_i T^2 h_i - \sum_{i=1}^{n} \lambda_i Th_i \|$$

$$+ \| \sum_{i=1}^{n} \lambda_i Th_i - g \|$$

$$\leqslant r.$$

Thus,

$$g \in H \text{ and } Tg \in G.$$

Since K_1 is minimal, $K_1 = G$.

But

$$\begin{aligned} D(G) = D(\overline{co}(T(H))) &= D(T(H)) \\ &= \sup\{\, \| Tx - Ty \| : x, y \in H \,\} \\ &\leqslant r < D(K_1). \end{aligned}$$

Thus, $D(K_1) > 0$ leads to a contradiction. Hence, $D(K_1) = 0$ and T has a fixed point, say p in K. It can be easily seen from (1) that p is unique.

Corollary 1 — If K be a convex subset of a Banach space X and $T : K \to K$ is continuous and satisfies (1), then T has a unique fixed point in K if any one of the following conditions is satisfied :

K is closed and bounded and X is a uniformly convex Banach space ...(2)

or

K is compact. ...(3)

PROOF : In either case, K has a normal structure.

§3. In this section, we present a fixed point theorem for densifying mapping on a strictly convex Banach space. We need the following preliminaries.

Definitions — Let A be a bounded set of a metric space X. We denote by $\alpha(A)$ the infinitum of all $\epsilon > 0$, such that A admits a finite convering consisting of subsets with diameter less than ϵ (Kuratowski 1971). A continuous mapping $T : X \to X$ is taken to be densifying, if for every bounded subset A of X such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

$\alpha(A)$ satisfies the following properties (Darbo 1967).

$0 \leqslant \alpha(A) \leqslant D(A)$ where $D(A)$ is the diameter of A. ...(i)

$\alpha(A) = 0 \Leftrightarrow A$ is precompact i.e. totally bounded. ...(ii)

$\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$ ...(iii)

$\alpha(\bar{A}) = 0 \Leftrightarrow \alpha(A) = 0$ where $\bar{A}$ is the closure of A. ...(iv)

The following known results will be used in our proof.

Theorem A (Furi and Vignoli 1969) — Let $T : K \to K$ be a densifying mapping defined on a closed, bounded, convex subset K of a Banach space X. Then, T has at least one fixed point.

Theorem B (Diaz and Metcalf 1969) — Let $T : X \to X$ be a continuous mapping of a metric space X into itself, such that

$F(T)$ is nonempty, where $F(T)$ is the set of fixed points of T ...(4)

for each $y \in F(x)$ with $y \notin F(T)$ and each $u \in F(T)$ one has

$$d(Ty, u) < d(y, u). \qquad ...(5)$$

Let $x_0 \in X$. Then, either $\{T^n x_0\}$ contains no convergent subsequence or $\lim_{n\to\infty} T^n x_0$ exists and belongs to $F(T)$.

We prove the following theorem.

Theorem 2 — Let $T : K \to K$ be a densifying mapping defined on a closed, convex, bounded subset K of a strictly convex Banach space X, such that T satisfies (1). Then for each $x \in K$, the sequence $\{S^n(x)\}$ defined by $S : K \to K$, where

$$S = \lambda_0 I + \lambda_1 T + \dots + \lambda_n T^n, \ \lambda_i \geqslant 0, \ \lambda_1 > 0, \ \sum_{i=0}^{n} \lambda_i = 1,$$

converges to a fixed point of T.

Proof : It is easy to see that S is a densifying mapping of K into itself. Moreover, $F(T)$ and $F(S)$ coincide, and by Theorem A, $F(T)$ and therefore, $F(S)$ is nonempty.

For

$$x \in K, \text{ let } A = \bigcup_{n=0}^{\infty} S^n(x).$$

Then

$$SA = \bigcup_{n=1}^{\infty} S^n(x) \subset A.$$ Also, since S is continuous, we have

$$S(\bar{A}) \subset \overline{S(A)} \subset \bar{A},$$ i.e., $\bar{A}$ is invariant under S also.

Now we shall prove that $\bar{A}$ is compact. It is sufficient to show that $\alpha(A) = 0$, since in a complete metric space (and, therefore, in a Banach space), the precompact sets are also relatively compact. Suppose

$$\alpha(A) > 0, \quad A = S(A) \cup \{x\}$$

$$\alpha(A) = \max \{\alpha(S(A)), \alpha\{x\}\}$$

$$= \max \{\alpha(S(A)), 0\} = \alpha S(A).$$

But this contradicts that S is densifying ; hence $\alpha(A) = 0$.

Thus, $\bar{A}$ is compact. Hence, the sequence of iterates has a convergent subsequence. Also, X strictly convex and condition (1) imply condition (5) of theorem B. Hence, by theorem B, $\{S^n x\}$ converges to a fixed point of T.

Remarks : Results similar to those of Browder and Petryshyn (1966), Belluce and Kirk (1966), Diaz and Metcalf (1969), Kirk (1971), Petryshyn (1971) and Petryshyn and Williamson (1973) which hold for nonexpansive mappings, hold for a mapping satisfying (1).

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