TRANSITION THEORY OF CREEP OF COMPOSITE SPHERICAL SHELLS UNDER UNIFORM INTERNAL PRESSURE

by SURESH HULSURKAR, *Department of Mathematics, Indian Institute of Technology, Kharagpur*

(*Received* 21 *May* 1977)

The creep stresses in a composite spherical shell under uniform internal pressure have been derived using Seth's transition theory. The expressions obtained are valid for compressible materials and no assumptions have been made about the creep laws. However, in special cases, the results obtained are identical with those resulting from the usual creep theory.

1. INTRODUCTION

The transition theory developed by Seth (1962, 1964, 1970, 1972) has been successfully applied to a large number of problems of plastic and creep deformations. A novel feature of the transition theory is its ability to give a uniform treatment of plastic and creep deformations without making ad hoc assumptions like yield criteria or creep laws. In this paper, the transition theory is applied to the creep of composite spheres under uniform internal pressure. The transitional creep stresses have been obtained which are valid for compressible materials. The creep stresses for incompressible materials can be derived by making certain parameter $c \to 0$. The expressions for creep stresses have been obtained in a very simple form for a special case in which the two materials of the composite sphere have the same creep index, but different creep constants. The reason for considering such a situation is based on an observation made by Odqvist and Mellgren (1958), that many materials have such a property under identical creep conditions. Our solutions in this special case conform to the results obtained by Mukherji (1964) based on the usual creep theory.

2. THE EQUATION OF EQUILIBRIUM AND THE TRANSITION POINTS

We consider a spherical shell made of two different isotropic materials under uniform internal pressure at some constant temperature under steady-state creep conditions.

The symmetry of the structure and the loading suggests (Seth 1972) that the deformations should be of the following type in spherical coordinates (r, θ, ϕ) :

$$u = r(1 - \beta), \; v = 0, \; w = 0 \qquad ...(2.1)$$

where

$$\beta = \beta(r).$$

The components of generalized strain are (Seth 1972)

$$\left.\begin{aligned} e_{rr} &= \frac{1}{n^m}\{1-(r\beta'+\beta)^n\}^m \\ e_{\theta\theta} &= e_{\phi\phi} = \frac{1}{n^m}(1-\beta^n)^m \\ e_{r\theta} &= e_{\theta\phi} = e_{r\phi} = 0 \end{aligned}\right\} \qquad \ldots(2.2)$$

where dash denotes differentiation with respect to r.

Substituting these values of the strains in the stress-strain relation

$$T_{ij} = \lambda \triangle \delta_{ij} + 2\mu e_{ij}$$

with the usual notations, we get

$$\left.\begin{aligned} T_{rr} &= \lambda\triangle + \frac{2\mu}{n^m}\{1-(r\beta'+\beta)^n\}^m \\ T_{\theta\theta} &= T_{\phi\phi} = \lambda\triangle + \frac{2\mu}{n^m}(1-\beta^n)^m \\ T_{r\theta} &= T_{\theta\phi} = T_{r\phi} = 0. \end{aligned}\right\} \qquad \ldots(2.3)$$

where

$$\triangle = \frac{1}{n^m}[2(1-\beta^n)^m + \{1-(r\beta'+\beta)^n\}^m].$$

The equation of equilibrium to be satisfied is

$$\frac{\partial T_{rr}}{\partial r} + \frac{2(T_{rr}-T_{\theta\theta})}{r} = 0 \qquad \ldots(2.4)$$

which on using eqn. (2.3) leads to

$$\{1-(r\beta'+\beta)^n\}^{m-1}(r\beta'+\beta)^{n-1}(r\beta''+2\beta') + 2(1-c)(1-\beta^n)^{m-1}\beta^{n-1}\beta'$$
$$-\frac{2c}{mnr}[\{1-(r\beta'+\beta)^n\}^m - (1-\beta^n)^m] = 0 \qquad \ldots(2.5)$$

after putting $c = \dfrac{2\mu}{(\lambda+2\mu)}$.

Putting $r\beta' = \beta P$, we get

$$\{1-\beta^n(P+1)^n\}^{m-1}P(P+1)^{n-1}\beta\frac{dP}{d\beta} + P(P+1)^n\{1-\beta^n(P+1)^n\}^{m-1}$$
$$+2P(1-c)(1-\beta^n)^{m-1} - \frac{2c}{mn\beta^n}[\{1-\beta^n(P+1)^n\}^m - (1-\beta^n)^m] = 0.$$
$$\ldots(2.6)$$

The critical point of interest for eqn. (2.6) is $P = -1$. Other critical points do not give anything of interest, as shown by Hulsurkar (1966).

3. The Transition Function and the Creep Law

The creep stresses are calculated through the transition function R_2 defined by

$$R_2 = T_{\theta\theta} - T_{rr} \qquad \text{...(3.1)}$$

which was used by Hulsurkar (1966).

From eqns. (3.7), (2.3) and using $r\beta' = \beta P$, we get

$$\dot{R_2} = \frac{2\mu}{n^m} [(1 - \beta^n)^m - \{1 - \beta^n(P + 1)^n\}^m]. \qquad \text{...(3.2)}$$

Taking logarithmic differentiation of eqn. (3.2) with respect to β and using eqn. (2.6) for the expression of $dP/d\beta$,

$$\frac{d \log R_2}{d \log \beta} = -2c \frac{d \log r}{d\beta} - \frac{(3 - 2c)\, mn\beta^n(1 - \beta^n)^{m-1}}{\{1 - \beta^n(P + 1)^n\}^m - (1 - \beta^n)^m}.$$

This gives

$$\log\left(\frac{R_2}{A_0}\right) = -2c\,.\log r + (3 - 2c) \log\{1 - (1 - \beta^n)^m\},$$

as $P \to -1$, where A_0 is a constant of integration. Therefore, we have

$$T_{\theta\theta} - T_{rr} = A_0 r^{-2c}\{1 - (1 - \beta^n)^m\}^{3-2c}.$$

Since $\beta \to \left(\frac{\beta_0}{r}\right)$ as $P \to -1$, where β_0 is constant, we get

$$T_{\theta\theta} - T_{rr} = A_0 r^{-2c}\{1 - (1 - B_0 r^{-n})^m\}^{3-2c} \qquad \text{...(3.3)}$$

where A_0, B_0 are constants.

Equation (3.3) will lead to a very general form of solutions in creep. For more information, the reader is referred to Seth (1972). For our problem, we shall take $m = 1$. Then eqn. (3.3) gives

$$T_{\theta\theta} - T_{rr} = Ar^{-3n+2c(n-1)} \qquad \text{...(3.4)}$$

where A is constant. Combining eqns. (3.4) and (2.4), we get

$$T_{rr} = \frac{2A}{-3n + 2c(n - 1)} r^{-3n+2c(n-1)} + B \qquad \text{...(3.5a)}$$

where B is a constant of integration.

From eqns. (3.4) and (3.5a)

$$T_{\theta\theta} = \frac{A\{-3n + 2 + 2c(n-1)\}}{-3n + 2c(n-1)} r^{-3n+2c(n-1)} + B. \qquad ...(3.5b)$$

The stress-strain relations can be derived as follows (Hulsurkar 1966). When the creep sets in, the relation to be used is

$$\dot{e}_{ij} = \frac{3}{2} \lambda_1 T'_{ij} \qquad ...(3.6a)$$

where $\dot{e}_{ij}$ is the rate of strain tensor e_{ij} with respect to some suitable flow parameter and T'_{ij} is the stress deviator.

From eqn. (2.2)

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\dot{\beta}. \qquad ...(3.6b)$$

If ϵ_{ij} denotes the Cauchy strain tensor, then

$$\dot{\epsilon}_{\theta\theta} = -\dot{\beta}. \qquad ...(3.6c)$$

Using eqns. (3.2). (3.6a), (3.6b), (3.6c), as $P \to -1$, we get

$$\dot{\epsilon}_{\theta\theta} = K(T_{\theta\theta} - T_{rr})^{(1/n)-1} T'_{\theta\theta}. \qquad ...(3.6d)$$

Now since the form in eqn. (3.6a) is to be valid, we should have

$$\dot{\epsilon}_{ij} = K(T_{\theta\theta} - T_{rr})^{(1/n)-1} T'_{ij}. \qquad ...(3.6)$$

If the flow parameter is taken as time t, the expression in eqn. (3.6) is the same as that which has been used by Mukherji (1964).

4. The Transitional Creep Stresses

In creep of composite sphere under internal pressure, we assume that the strain measure exponents are n_1 and n_2, and the creep constants are K_1 and K_2 respectively.

Hence, the creep stresses are given by

$$\left.\begin{aligned}
(T_{rr})_1 &= \frac{2A_1}{-3n_1 + 2c_1(n_1-1)} r^{-3n_1+2c_1(n_1-1)} + B_1 \\
(T_{\theta\theta})_1 &= \frac{A_1\{3n_1 - 2 - 2c_1(n_1-1)\}}{3n_1 - 2c_1(n_1-1)} r^{-3n_1+2c_1(n_1-1)} + B_1 \\
\text{and} \qquad & \\
(T_{rr})_2 &= \frac{2A_2}{-3n_2 + 2c_2(n_2-1)} r^{-3n_2+2c_2(n_2-1)} + B_2 \\
(T_{\theta\theta})_2 &= \frac{A_2\{3n_2 - 2 - 2c_2(n_2-1)\}}{3n_2 - 2c_2(n_2-1)} r^{-3n_2+2c_2(n_2-1)} + B_2.
\end{aligned}\right\} \qquad ...(4.1)$$

If the internal, external and intermediate radii are a, b and d respectively of the composite sphere under internal pressure p, the boundary conditions are

(i) $(T_{rr})_1 = -p$ at $r = a$

(ii) $(T_{rr})_2 = 0$ at $r = b$

and

(iii) $(T_{rr})_1 = (T_{rr})_2$ at $r = d$

(iv) $(\dot{\epsilon}_{rr})_1 = (\dot{\epsilon}_{rr})_2$ at $r = d$. ...(4.2)

From eqns. (3.4) and (3.6), the boundary condition (4.2iv) gives

$$K_1(A_1)^{1/n_1} d^{-2c_1(1-(1/n_1))} = K_2(A_2)^{1/n_2} d^{-2c_2(1-(1/n_2))}$$
$$= \alpha$$

which leads to

$$A_1 = \left(\frac{\alpha}{K_1}\right)^{n_1} d^{-2c_1(n_1-1)}, \quad A_2 = \left(\frac{\alpha}{K_2}\right)^{n_2} d^{-2c_2(n_2-1)}. \qquad \text{...(4.3)}$$

Using the boundary conditions (4.2i), (4.2ii) and eqns. (4.3), we get

$$\text{(i)} \quad (T_{rr})_1 = \frac{2\left(\frac{\alpha}{K_1}\right)^{n_1} d^{-2c_1(n_1-1)}}{-3n_1 + 2c_1(n_1-1)} \{r^{-3n_1+2c_1(n_1-1)} - a^{-3n_1+2c_1(n_1-1)}\} - p$$

$$\text{(ii)} \quad (T_{rr})_2 = \frac{2\left(\frac{\alpha}{K_2}\right)^{n_2} d^{-2c_2(n_2-1)}}{-3n_2 + 2c_2(n_2-1)} \{r^{-3n_2+2c_2(n_2-1)} - b^{-3n_2+2c_2(n_2-1)}\}$$

$$\text{(iii)} \quad (T_{\theta\theta})_1 = \frac{\left(\frac{\alpha}{K_1}\right)^{n_1} d^{-2c_1(n_1-1)}}{3n_1 - 2c_1(n_1-1)} [\{3n_1 - 2 - 2c_1(n_1-1)\} \times r^{-3n_1+2c_1(n_1-1)} + 2a^{-3n_1+2c_1(n_1-1)}] - p$$

$$\text{(iv)} \quad (T_{\theta\theta})_2 = \frac{\left(\frac{\alpha}{K_2}\right)^{n_2} d^{-2c_2(n_2-1)}}{3n_2 - 2c_2(n_2-1)} [\{3n_2 - 2 - 2c_2(n_2-1)\} \times r^{-3n_2+2c_2(n_2-1)} + 2b^{-3n_2+2c_2(n_2-1)}] \qquad \text{...(4.4)}$$

where α is given by

$$\frac{2\left(\frac{\alpha}{K_1}\right)^{n_1} d^{-2c_1(n_1-1)}}{-3n_1 + 2c_1(n_1-1)} \{d^{-3n_1+2c_1(n_1-1)} - a^{-3n_1+2c_1(n_1-1)}\} - p$$
$$= \frac{2\left(\frac{\alpha}{K_2}\right)^{n_2} d^{-2c_2(n_2-1)}}{-3n_2 + 2c_2(n_2-1)} \{d^{-3n_2+2c_2(n_2-1)} - b^{-3n_2+2c_2(n_2-1)}\}. \qquad \text{...(4.5)}$$

Equations (4.4) together with eqn. (4.5) determines the transitional stresses in creep. For those materials which are compressible in creep, eqn. (4.4) gives the creep stresses. If materials are incompressible in creep, we make $c_1 \to 0$ and $c_2 \to 0$. This gives

$$\left.\begin{aligned}
(T_{rr})_1 &= \frac{2\left(\frac{\alpha}{K_1}\right)^{n_1}}{3n_1}(a^{-3n_1} - r^{-3n_1}) - p \\
(T_{rr})_2 &= \frac{2\left(\frac{\alpha}{K_2}\right)^{n_2}}{3n_2}(b^{-3n_2} - r^{-3n_2}) \\
(T_{\theta\theta})_1 &= \frac{\left(\frac{\alpha}{K_1}\right)^{n_1}}{3n_1}\{(3n_1 - 2)\, r^{-3n_1} + 2a^{-3n_1}\} - p \\
(T_{\theta\theta})_2 &= \frac{\left(\frac{\alpha}{K_2}\right)^{n_2}}{3n_2}\{(3n_2 - 2)\, r^{-3n_2} + 2b^{-3n_2}\}
\end{aligned}\right\} \quad \ldots(4.6)$$

where α is given by

$$\frac{2}{3n_1}\left(\frac{\alpha}{K_1}\right)^{n_1}(a^{-3n_1} - d^{-3n_1}) - p = \frac{2}{3n_2}\left(\frac{\alpha}{K_2}\right)^{n_2}(b^{-3n_2} - d^{-3n_2}).$$

5. A Special Case

Now under identical creep conditions, more than one metal may have the same creep index (which means in our case the strain measure n as $n = (1/s)$, where s is the creep index), while the creep constant may be different (Odqvist and Mellgren 1958). For two such materials, which have been used for composite shell, we can assume $n_1 = n_2 = n$ and we get the transitional stresses in creep as

$$\text{(i)} \quad (T_{rr})_1 = \frac{2\left(\frac{\alpha}{K_1}\right)^{n} d^{-2c_1(n-1)}}{-3n + 2c_1(n-1)}\{r^{-3n+2c_1(n-1)} - a^{-3n+2c_1(n-1)}\} - p$$

$$\text{(ii)} \quad (T_{rr})_2 = \frac{2\left(\frac{\alpha}{K_2}\right)^{n} d^{-2c_2(n-1)}}{-3n + 2c_2(n-1)}\{r^{-3n+2c_2(n-1)} - b^{-3n+2c_2(n-1)}\}$$

$$\text{(iii)} \quad (T_{\theta\theta})_1 = \frac{\left(\frac{\alpha}{K_1}\right)^{n} d^{-2c_1(n-1)}}{3n - 2c_1(n-1)}[\{3n - 2 - 2c_1(n-1)\} \times r^{-3n+2c_1(n-1)} + 2a^{-3n+2c_1(n-1)}] - p$$

$$\text{(iv)} \quad (T_{\theta\theta})_2 = \frac{\left(\frac{\alpha}{K_2}\right)^{n} d^{-2c_2(n-1)}}{3n - 2c_2(n-1)}[\{3n - 2 - 2c_2(n-1)\} \times r^{-3n+2c_2(n-1)} + 2b^{-3n+2c_2(n-1)}] \quad \ldots(5.1)$$

where α is given by

$$2\alpha^n\left[\frac{d^{-2c_1(n-1)}}{K_1^n\{3n-2c_1(n-1)\}}\{a^{-3n+2c_1(n-1)}-d^{-3n+2c_1(n-1)}\}\right.$$
$$\left.-\frac{d^{-2c_2(n-1)}}{K_2^n\{3n-2c_2(n-1)\}}\{b^{-3n+2c_2(n-1)}-d^{-3n+2c_2(n-1)}\}\right]=p.$$

The creep stresses for incompressible materials is obtained by letting $c_1 \to 0$ and $c_2 \to 0$. Thus, from eqns. (5.1), we get the creep stresses as

$$\text{(i)}\quad (T_{rr})_1 = p\frac{\left(\frac{K_2}{K_1}\right)^n(d^{-3n}-r^{-3n})+(b^{-3n}-d^{-3n})}{\left(\frac{K_2}{K_1}\right)^n(a^{-3n}-d^{-3n})-(b^{-3n}-d^{-3n})}$$

$$\text{(ii)}\quad (T_{rr})_2 = p\frac{(b^{-3n}-r^{-3n})}{\left(\frac{K_2}{K_1}\right)^n(a^{-3n}-d^{-3n})-(b^{-3n}-d^{-3n})}$$

$$\text{(iii)}\quad (T_{\theta\theta})_1 = p\frac{\left(\frac{K_2}{K_1}\right)^n[\{(\tfrac{3}{2})\,n-1\}\,r^{-3n}+d^{-3n}]+(b^{-3n}-d^{-3n})}{\left(\frac{K_2}{K_1}\right)^n(a^{-3n}-d^{-3n})-(b^{-3n}-d^{-3n})}$$

$$\text{(iv)}\quad (T_{\theta\theta})_2 = p\frac{\{(\tfrac{3}{2})\,n-1\}\,r^{-3n}+b^{-3n}}{\left(\frac{K_2}{K_1}\right)^n(a^{-3n}-d^{-3n})-(b^{-3n}-d^{-3n})}. \qquad \ldots(5.2)$$

Equations (4.6) and (5.2) are the same as those derived by Mukherjee (1964), who used the compressibility condition and the usual creep laws.

The discontinuity in $T_{\theta\theta}$ at $r = d$ is given by

$$\{(T_{\theta\theta})_1-(T_{\theta\theta})_2\}_{r=d} = p\frac{(\tfrac{3}{2})\,n\left\{\left(\frac{K_2}{K_1}\right)^n-1\right\}d^{-3n}}{\left(\frac{K_2}{K_1}\right)^n(a^{-3n}-d^{-3n})-(b^{-3n}-d^{-3n})}.$$

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