ON VISCO-ELASTIC FREE CONVECTION BOUNDARY LAYER FLOW PAST AN INFINITE PLATE WITH CONSTANT SUCTION

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The behaviour of unsteady free convective flow of an elastico-viscous fluid past an infinite, porous plate with constant suction is studied. It is assumed that the plate temperature oscillates in magnitude about a constant mean but not in direction. The numerical values of M_r and M_i, the real and imaginary parts of the fluctuations of the velocity, are tabulated and some variations are represented graphically. Graphical representation has also been made for the phase angle ϕ and the other terms occurring in the calculations for different values of the frequency of oscillations. It is concluded that the velocity distribution inside the boundary layer region lags behind the wall fluctuation by a certain angle ϕ.

NOTATIONS

c_p' = specific heat at constant pressure

f_x = acceleration due to gravity

k_0 = elastic constant

M_r, M_i = fluctuating parts of the velocity for $P \neq 1$

P = Prandtl number

T' = temperature in the boundary layer

T_∞' = temperature of fluid far away from the plate

T_w' = plate temperature

t' = time

(u', v') = velocity components

v_0 = suction velocity

x, y = co-ordinate axes along and normal to the plate

ρ' = density

β = coefficient of volume expansion

η_0 = viscosity

λ' = thermal conductivity

ν = kinematic viscosity
ω = frequency
ϵ = constant
ϕ = phase angle.

1. Introduction

Many papers have been published on the theory of laminar boundary layers in unsteady flow. Nanda and Sharma (1963) studied free convection laminar boundary layer flows from a vertical flat plate when the plate temperature oscillates in time about a constant non-zero mean. Free convection flow in the boundary layer from a semi-infinite vertical plate in which the mean surface temperature varies as a function of the distance from the leading edge of the plate was studied by Mishra and Panda (1976). Free convective flow past an infinite porous plate with suction varying as $t^{(-1/2)}$ was studied by Nanda and Sharma (1962) in the case of ordinary fluids. Analysis of the effects of a variable suction and the horizontal magnetic field on the free convective flow past an infinite vertical, porous plate was carried out by Soundalgekar (1972). The effect of unsteady flow in the magnitude of surface temperature on the free convection laminar velocity and thermal boundary layers on a flat plate was studied by Lal (1975). He obtained the solution when the fluctuation is an exponentially decreasing function of time and found that the temperature distribution varies as an oscillatory function of time for high frequency of fluctuations. Lal (1969) obtained solutions for unsteady free convection flow of laminar power law fluids past a porous vertical wall. He found that if the wall temperature and suction velocity vary as $t^{-1/2}$, the similarity solution is possible.

The object of the present study is to discuss the effects of the time-dependent plate temperature on the free convective flow of a visco-elastic fluid in the presence of constant suction. The temperature profile has been discussed by Pop (1968). The transient velocity, skin friction amplitude and skin friction phase have been discussed by Soundalgekar (1971) for large frequency. In the present paper, we have considered small values of P and ω with k varying from 0 to 1. The phase-angle $|\phi|$ and some variations are calculated and tabulated and are also presented graphically. Separate expressions for M_r and M_i are obtained. Phase-angle variations with respect to ω are tabulated. To the best of the author's knowledge, the above calculations, and tabular data are being presented for the first time. The data are mathematically analysed in Section 2 and a general conclusion is given in Section 3.

2. Mathematical Analysis

The basic equations are

$$\frac{\partial v'}{\partial y'} = 0 \qquad \ldots(1)$$

$$\rho' \frac{\partial u'}{\partial t'} + \rho' v' \frac{\partial u'}{\partial y'} = \rho' f_x \beta(T' - T'_\infty) + \eta_0 \frac{\partial^2 u'}{\partial y'^2}$$

$$- k_0 \left(\frac{\partial^3 u'}{\partial y'^2 \partial t'} + v' \frac{\partial^3 u'}{\partial y'^3} - 3 \frac{\partial u'}{\partial y'} \frac{\partial^2 v'}{\partial y'^2} - 2 \frac{\partial v'}{\partial y'} \frac{\partial^2 u'}{\partial y'^2} \right) \quad \ldots(2)$$

$$\rho' \frac{\partial v'}{\partial t'} + \rho' v' \frac{\partial v'}{\partial y'} = - \frac{\partial p'}{\partial y'} + 2\eta_0 \frac{\partial^2 v'}{\partial y'^2}$$

$$- 2k_0 \left(\frac{\partial^3 v'}{\partial y'^2 \partial t'} + v' \frac{\partial^3 v'}{\partial y'^3} - 3 \frac{\partial v'}{\partial y'} \frac{\partial^2 v'}{\partial y'^2} \right) \quad \ldots(3)$$

$$\rho' c'_p \left(\frac{\partial T}{\partial t'} + v' \frac{\partial T'}{\partial y'} \right) = \lambda' \frac{\partial^2 T'}{\partial y'^2}. \quad \ldots(4)$$

The origin is taken at any point of a flat, vertical, porous, infinite plate. The x'-axis is chosen along the plate vertically upwards and the y'-axis perpendicular to it. u' is the velocity in the x'-direction and v', the velocity normal to the plate; t' is the time variable; η_0, the limiting viscosity at small rates of shear; and k_0, the elastic constant. Also, f_x is the acceleration due to gravity; β, the coefficient of volume expansion; c'_p, the specific heat at constant pressure; λ', the thermal conductivity; T', the temperature in the boundary layer; and T'_∞, the temperature far away from the plate. In eqn. (4), terms representing viscous and elastic dissipation are assumed to be neglected.

From (1), for constant suction velocity v_0,

$$v' = - v_0. \quad \ldots(5)$$

Applying (5) in eqns. (2) and (3), we have

$$\frac{\partial u'}{\partial t'} - v_0 \frac{\partial u'}{\partial y'} = f_x \beta(T' - T'_\infty) + \nu \frac{\partial^2 u'}{\partial y'^2}$$

$$- k^* \left[\frac{\partial^3 u'}{\partial y'^2 \partial t} - v_0 \frac{\partial^3 u'}{\partial y'^3} \right] \quad \ldots(6)$$

$$\frac{\partial v'}{\partial t'} = - \frac{1}{\rho'} \frac{\partial p'}{\partial y'} \quad \ldots(7)$$

where

$$\nu = \eta_0/\rho' \text{ and } k^* = k_0/\rho'.$$

The boundary conditions are :

$$\left.\begin{array}{lll} u' = 0, \ T' = T'_w(t) & \text{at } y' = 0 \\ u' \to 0, \ T' \to 0 & \text{as } y' \to \infty. \end{array}\right\} \quad \ldots(8)$$

We introduce non-dimensional quantities in eqns. (8)

$$\eta = y'v_0/\nu, \quad t' = v_0^2 t'/4\nu, \quad u = u'/Gv_0$$

$$k = k^* v_0^2/\nu^2, \quad T = \frac{T' - T'_\infty}{T'_w - T'_\infty} \qquad \ldots(9)$$

$$G = \nu f_x \beta(T'_w - T'_\infty)/v_0^3, \text{ the Grashoff number}$$

$$P = \eta_0 c'_p/\lambda', \text{ the Prandtl number.}$$

Equations (6) and (4) then reduce to

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \eta} - \tfrac{1}{4}\frac{\partial u}{\partial t} - k\left(\tfrac{1}{4}\frac{\partial^3 u}{\partial \eta^2 \partial t} - \frac{\partial^3 u}{\partial \eta^3}\right) = -T \qquad \ldots(10)$$

and

$$\frac{\partial^2 T}{\partial \eta^2} + P\frac{\partial T}{\partial \eta} - \frac{P}{4}\frac{\partial T}{\partial t} = 0. \qquad \ldots(11)$$

The boundary conditions (8) now reduce to

$$\left.\begin{array}{lll} u = 0, & T = T_w(t) & \text{at } \eta = 0 \\ u \to 0, & T \to 0 & \text{as } \eta \to \infty. \end{array}\right\} \qquad \ldots(12)$$

Let the temperature and the velocity in the neighbourhood of the plate be assumed to be (Pop 1968) :

$$T(\eta, t) = [1 - f_1(\eta)] + \epsilon e^{i\omega t}[1 - f_2(\eta)] \qquad \ldots(13)$$

and

$$u(\eta, t) = g_1(\eta) + \epsilon e^{i\omega t} g_2(\eta) \qquad \ldots(14)$$

respectively.

Substituting (13) in (11) and solving after comparing the harmonic terms, we have

$$f_1(\eta) = 1 - e^{-P\eta} \qquad \ldots(15)$$

$$f_2(\eta) = 1 - e^{-PH\eta} \qquad \ldots(16)$$

where

$$H = \tfrac{1}{2}\left\{1 + \left(1 + \frac{i\omega}{P}\right)^{\frac{1}{2}}\right\}$$

and f_1, f_2 satisfy the reduced boundary conditions

$$\left.\begin{array}{ll} f_1 = f_2 = 0 & \text{at } \eta = 0 \\ f_1 \to 1, f_2 \to 1 & \text{as } \eta \to \infty. \end{array}\right\} \qquad \ldots(17)$$

Substituting (13) and (14) in (10) and comparing harmonic terms, we get

$$kg_1''' + g_1'' + g_1' = f_1 - 1 \quad ...(18)$$

$$kg_2''' + (1 - \tfrac{1}{4}ki\omega)\, g_2'' + g_2' - \frac{iw}{4} g_2 = f_2 - 1 \quad ...(19)$$

where the corresponding boundary conditions now become

$$\left.\begin{aligned} g_1 = g_2 = 0 \quad & \text{at } \eta = 0 \\ g_1 = g_2 = 0 \quad & \text{as } \eta \to \infty. \end{aligned}\right\} \quad ...(20)$$

Equations (18) and (19) reduce to those governing the flow of a Newtonian fluid, if $k = 0$. Let us assume the solution in the forms (Soundalgekar 1971)

$$g_1 = g_{01} + kg_{11} + O(k^2) \quad ...(21)$$

$$g_2 = g_{02} + kg_{12} + O(k^2). \quad ...(22)$$

Inserting (21) and (22) in (18) and (19) and equating the coefficients of k, we obtain, after substituting for f_1 and f_2 respectively from (15) and (16),

$$g_{01}'' + g_{01}' = - e^{-P\eta} \quad ...(23)$$

$$g_{11}'' + g_{11}' = - g_{01}''' \quad ...(24)$$

$$g_{02}'' + g_{02}' - \frac{i\omega}{4} g_{02} = - e^{-PH\eta} \quad ...(25)$$

$$g_{12}'' + g_{12}' - \frac{i\omega}{4} g_{12} = - g_{02}''' + \frac{i\omega}{4} g_{02}''. \quad ...(26)$$

The correspoding boundary conditions on $g_{01}, g_{11}, g_{02}, g_{12}$ are

$$\left.\begin{aligned} g_{01} = g_{11} = g_{02} = g_{12} = 0 \quad & \text{at } \eta = 0 \\ g_{01}, g_{11}, g_{02}, g_{12} \to 0 \quad & \text{as } \eta \to \infty. \end{aligned}\right\} \quad ...(27)$$

On solving eqns. (23) — (26) which satisfy (27), the velocity field in the boundary layer is obtained, for $P \neq 1$, as

$$u(\eta, t) = \frac{e^{-\eta} - e^{-P\eta}}{P^2 - P} + k\left[\frac{P}{(P-1)^2}(e^{-\eta} - e^{-P\eta}) - \frac{\eta e^{-\eta}}{P^2 - P}\right]$$

$$+ \epsilon e^{i\omega t}\left[\frac{e^{-h\eta} - e^{-PH\eta}}{P^2H^2 - PH - \frac{i\omega}{4}}\right.$$

(*equation continued on p.* 234)

$$+ k \left\{ \frac{h^3 + \frac{i\omega h^2}{4}}{\left(P^2H^2 - PH - \frac{i\omega}{4}\right)(1 - 2h)} \eta e^{-h\eta} \right.$$

$$\left. + \frac{P^2H^2\left(\frac{i\omega}{4} + PH\right)}{\left(P^2H^2 - PH - \frac{i\omega}{4}\right)^2} (e^{-h\eta} - e^{-PH\eta}) \right\} \Bigg] \qquad \ldots(28)$$

where

$$h = \tfrac{1}{2}\{1 + (1 + i\omega)^{1/2}\}.$$

Hence, we may write

$$u(\eta, t) = \frac{e^{-\eta} - e^{-P\eta}}{P^2 - P} + k\left[\frac{P}{(P-1)^2}(e^{-\eta} - e^{-P\eta}) - \frac{\eta e^{-\eta}}{P^2 - P}\right]$$

$$+ \epsilon (M_r \cos \omega t - M_i \sin \omega t) \qquad \ldots(29)$$

where M_r, M_i are the fluctuating parts of the velocity when $P \neq 1$, and are given by

$$M_r + i M_i = \frac{e^{-h\eta} - e^{-PH\eta}}{P^2H^2 - PH - \frac{i\omega}{4}}$$

$$+ k \left\{ \frac{h^3 + i\omega h^2/4}{\left(P^2H^2 - PH - \frac{i\omega}{4}\right)(1 - 2h)} \eta e^{-h\eta} \right.$$

$$\left. + \frac{P^2H^2\left(\frac{i\omega}{4} + PH\right)}{\left(P^2H^2 - PH - \frac{i\omega}{4}\right)^2} (e^{-h\eta} - e^{-PH\eta}) \right\}. \qquad \ldots(30)$$

Separating the real and imaginary parts, we have

$$M_r = \frac{e^{-a\eta} - e^{-AP\eta}}{B^2 + b^2}\left[B \cos \frac{\omega\eta}{4} + b \sin \frac{\omega\eta}{4}\right]$$

$$+ \frac{k\eta e^{-a\eta}}{a_3^2 - a_4^2}\left[(a_1a_3 + a_2a_4)\cos\frac{\omega\eta}{4} + (a_2a_3 - a_1a_4)\sin\frac{\omega\eta}{4}\right]$$

$$+ k\,\frac{e^{-a\eta} - e^{-AP\eta}}{(B^2 + b^2)^2}\left[a_5 \cos\frac{\omega\eta}{4} + a_6 \sin\frac{\omega\eta}{4}\right] \qquad \ldots(31)$$

and

$$M_i = -\frac{e^{-a\eta} - e^{-AP\eta}}{B^2 + b^2}\left[b \cos\frac{\omega\eta}{4} + B \sin\frac{\omega\eta}{4}\right] +$$

(equation continued on p. 235)

$$+ k \frac{\eta e^{-a\eta}}{a_3^2 + a_4^2} \left[(a_2 a_3 - a_1 a_4) \cos \frac{\omega\eta}{4} - (a_1 a_3 + a_2 a_4) \sin \frac{\omega\eta}{4} \right]$$

$$+ k \frac{e^{-a\eta} - e^{-AP\eta}}{(B^2 + b^2)^2} \left[a_6 \cos \frac{\omega\eta}{4} - a_5 \sin \frac{\omega\eta}{4} \right] \qquad ...(32)$$

where

$$A = 1 + \frac{\omega^2}{16P^2}, \qquad a = 1 + \frac{\omega^2}{16}$$

$$B = A^2P^2 - AP - \frac{\omega^2}{16}, \qquad b = (AP - 1)\, \tfrac{1}{2}\omega$$

$$a_1 = a^3 - \frac{5}{16} a\omega^2, \qquad a_2 = \left(a^2 - \frac{\omega^2}{32}\right)\omega$$

$$a_3 = B(1 - 2a) + \frac{b\omega}{2}, \quad a_4 = b(1 - 2a) - \frac{B\omega}{2}$$

$$a_5 = (B^2 - b^2)\left(A^3P^3 - \frac{5}{16} AP\omega^2\right) + 2Bb\left(A^2P^2\omega - \frac{\omega^3}{32}\right)$$

$$a_5 = (B^2 - b^2)\left(A^2P^2\omega - \frac{\omega^3}{32}\right) - 2Bb\left(A^3P^3 - \frac{5}{16} AP\omega^2\right). \qquad ...(33)$$

From (31) and (32), we get

$$\left| M_r + iM_i \right| = \sqrt{M_r^2 + M_i^2}$$

$$= \frac{e^{-a\eta} - e^{-AP\eta}}{B^2 + b^2} \left[\left(B^2 + b^2 + 4Bb \cos \frac{\omega\eta}{4} \sin \frac{\omega\eta}{4} \right) \right.$$

$$+ \left(\frac{k\eta e^{-a\eta}}{a_3^2 + a_4^2} \cdot \frac{B^2 + b^2}{e^{-a\eta} - e^{-AP\eta}} \right)^2 \{(a_1 a_3 + a_2 a_4)^2 + (a_2 a_3 - a_1 a_4)^2\}$$

$$+ \frac{k^2}{(B^2 + b^2)^2} (a_5^2 + a_6^2)$$

$$+ \frac{2k\eta e^{-a\eta}}{a_3^2 + a_4^2} \cdot \frac{B^2 + b^2}{e^{-a\eta} - e^{-AP\eta}} \left\{ B(a_1 a_3 + a_2 a_4) \right.$$

$$\left. + b(a_1 a_3 + a_2 a_4) \sin \frac{\omega\eta}{2} - b(a_2 a_3 - a_1 a_4) \cos \frac{\omega\eta}{2} \right\}$$

$$+ 2k^2 \frac{\eta e^{-a\eta}}{a_3^2 + a_4^2} \cdot \frac{1}{e^{-a\eta} - e^{-AP\eta}} \{a_5(a_1 a_3 + a_2 a_4) + a_6(a_2 a_3 - a_1 a_4)\}$$

$$\left. + 2\frac{k}{B^2 + b^2} \left\{ a_5 B + a_6 b \sin \frac{\omega\eta}{2} - a_5 b \cos \frac{\omega\eta}{2} \right\} \right]^{1/2}. \qquad ...(34)$$

Let P, ω, k be constants, A = constant, B = constant, b = constant, and a_i $(i = 1, 2, \ldots, 6)$ = constant. Then from (31), differentiating M_r with respect to η and making $\frac{\partial M_r}{\partial \eta} = 0$ for maximum or minimum values of M_r, we have, for different values of k, the points where M_r is maximum.

TABLE I

Real part M_r, when $P = 0.1$, $\omega = 0.2$.

$\eta \backslash k$	0	0.05	0.5	1
0	0	0	0	0
0.5	1.7860	1.8782	2.7083	3.6306
1	2.2636	2.4128	3.7563	5.2490
2	3.5820	3.6628	4.3902	5.1985
4	3.2835	2.3053	3.5019	3.7203
5	2.9651	2.9750	3.0646	3.1641

TABLE II

k	0	0.05	0.5	1
η	2.8	2.68	1.88	1.59

TABLE III

Imaginary part M_i, $P = 0.1$, $\omega = 0.2$.

$\eta \backslash k$	0	0.05	0.5	1
0	0	0	0	0
0.5	− 1.3681	− 1.4278	− 1.9651	− 2.5621
1	− 1.6666	− 1.7611	− 2.6118	− 3.5570
2	− 2.8060	− 2.8557	− 3.3035	− 3.8010
4	− 2.5621	− 2.5720	− 2.6616	− 2.7611
5	− 2.3184	− 2.3233	− 2.3681	− 2.4178

TABLE IV

Points where $|M_i|$ is maximum for different values of k.

k	0	0.05	0.5	1
η	3.32	3.08	2.28	1.9

TABLE V

Values of $|\phi|$ *(in deg.) for* $P = 0.1$, $\omega = 0.2$.

$\eta \diagdown k$	0	0.05	0.5	1
0.5	37.45	37.23	35.96	35.20
1	36.36	36.12	34.81	34.11
2	38.07	37.94	36.98	36.16
4	37.96	37.89	37.23	36.58
5	38.02	37.99	37.70	37.39

TABLE VI

$|M_r + iM_i| = \sqrt{M_r^2 + M_i^2} = |M|$, $P = 0.1$, $\omega = 0.2$.

$\eta \diagdown k$	0	0.05	0.5	1
0	0	0	0	0
0.5	2.280	2.360	3.360	4.440
1	2.860	2.941	4.604	6.325
2	4.561	4.623	5.499	6.440
4	4.201	4.201	4.421	4.582
5	3.780	3.780	3.842	4.000

TABLE VII

Variation of M_r, M_i *and* $|\phi|$ *with respect to* ω *for* $k = 0$, $P = 0.1$ *and* $\eta = 1$.

w	0	0.1	0.2	0.4	0.5	0.8	1.0
M_r	5.9629	2.2922	2.2636	1.0917	0.8223	0.3300	0.1211
M_i	0	– 2.19	– 2.21	– 1.32	– 1.01	– 0.412	– 0.19
ϕ (deg.)	0	43.33	44.46	50.46	50.90	51.30	57.74

3. CONCLUSIONS

The unsteady free convection flow of an elastico-viscous fluid past an infinite, porous plate with constant suction has been studied, when the plate temperature oscillates in magnitude about a constant mean.

The values of M_r for different values of k, considering P (= 0.1) and ω (= 0.2) as constants are tabulated in Table I and represented graphically in Fig. 1. We observe that for large values of k, M_r is stiffer and then slopes away. It is seen that for $k = 1$, M_r is changing appreciably in the range $1 < \eta < 2$ and then decreases as η increases up to 5. For $k = 0.5$, M_r has its maximum value at $\eta = 1.88$ and for $k = 0$, and 0.05, M_r changes its values in the range $2.5 < \eta < 3$ (Table II).

In Fig. 2, M_i is plotted and it is observed that M_i is negative throughout. The values of M_i are tabulated in Table III.

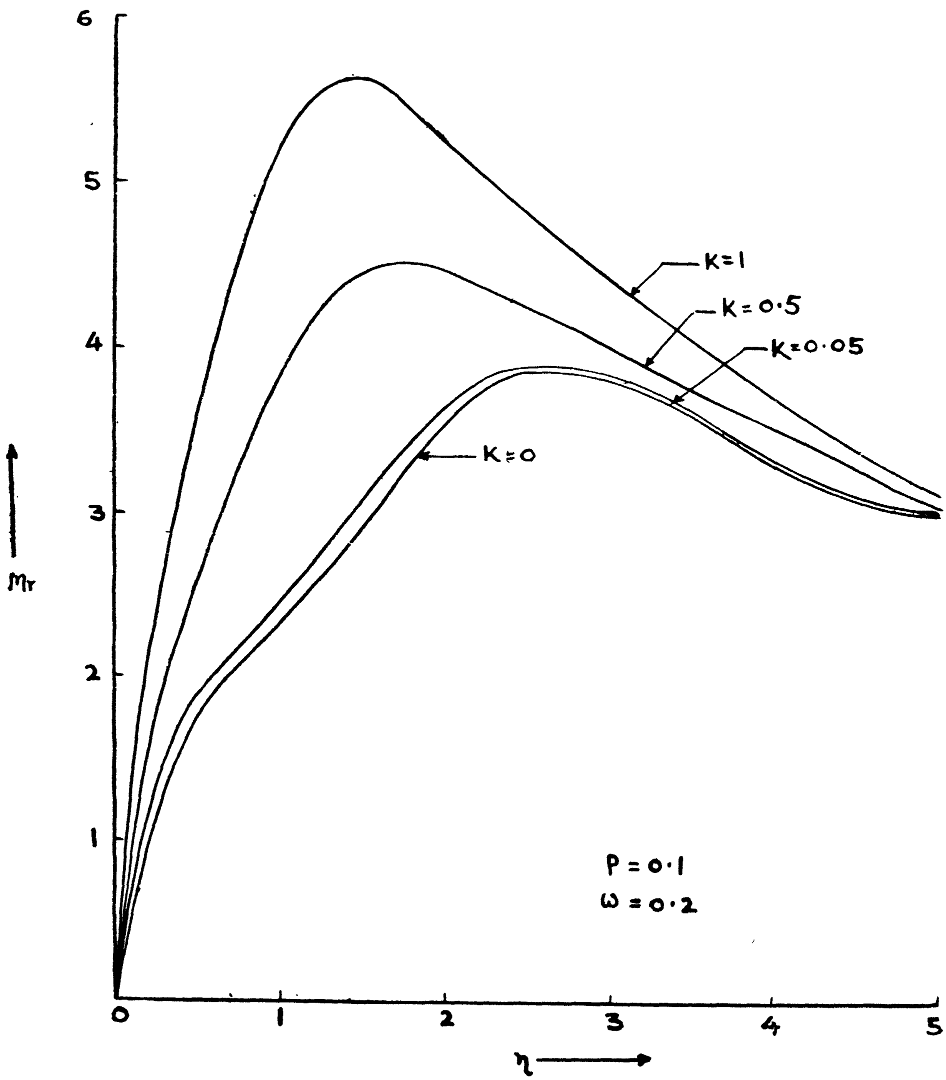

FIG. 1. The real part M_r for different k.

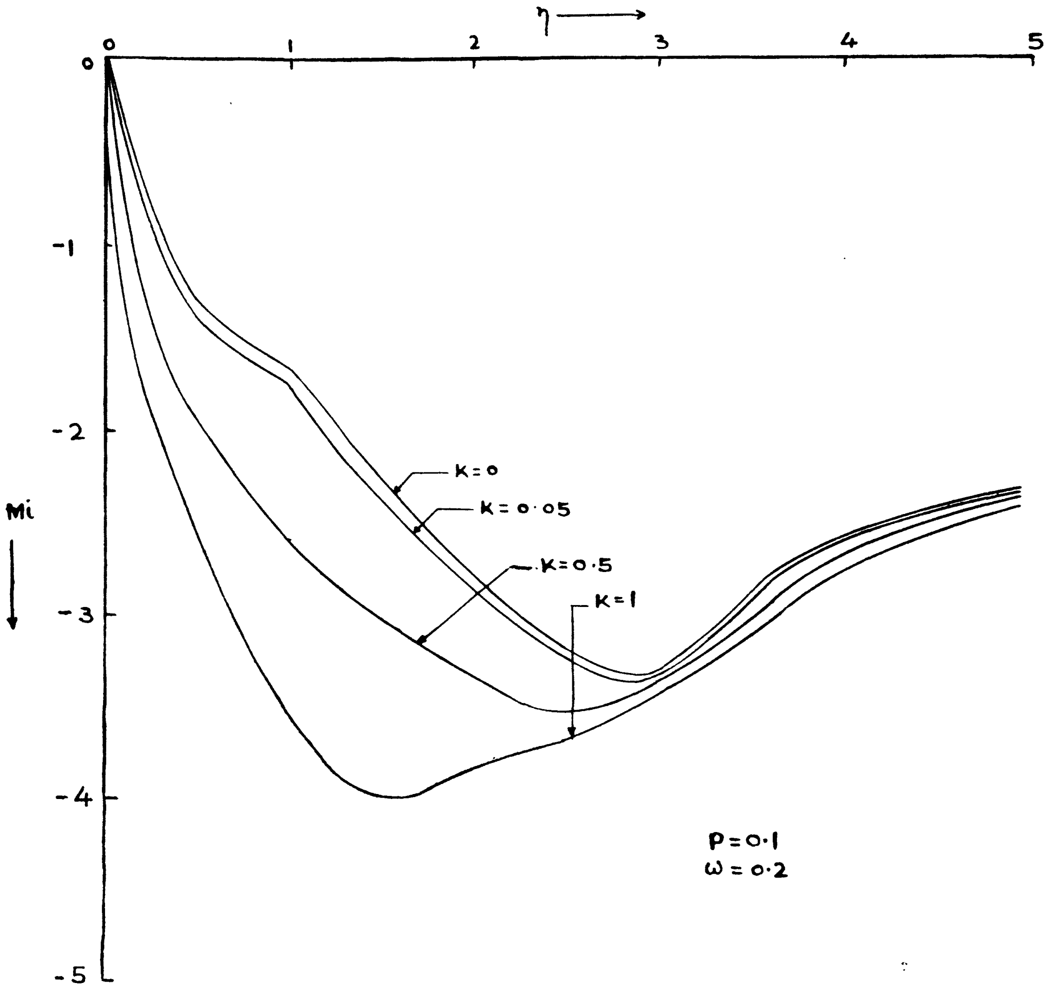

FIG. 2. The imaginary part M_i for different k.

In the velocity distribution eqn. (29), the coefficient of ϵ is

$$(M_r \cos \omega t - M_i \sin \omega t) = \sqrt{M_r^2 + M_i^2} \cos (\omega t + \phi).$$

This term lags or leads over fluctuation by an angle ϕ. In our case, ϕ is negative. The values of ϕ for different k values are given in Table V. The ratio M_i/M_r at $\eta = 0$ is not possible for k values other than 0. In Fig. 3, $| \phi |$ is plotted against η and the graphs are interesting. In each case, it shows an appreciable change

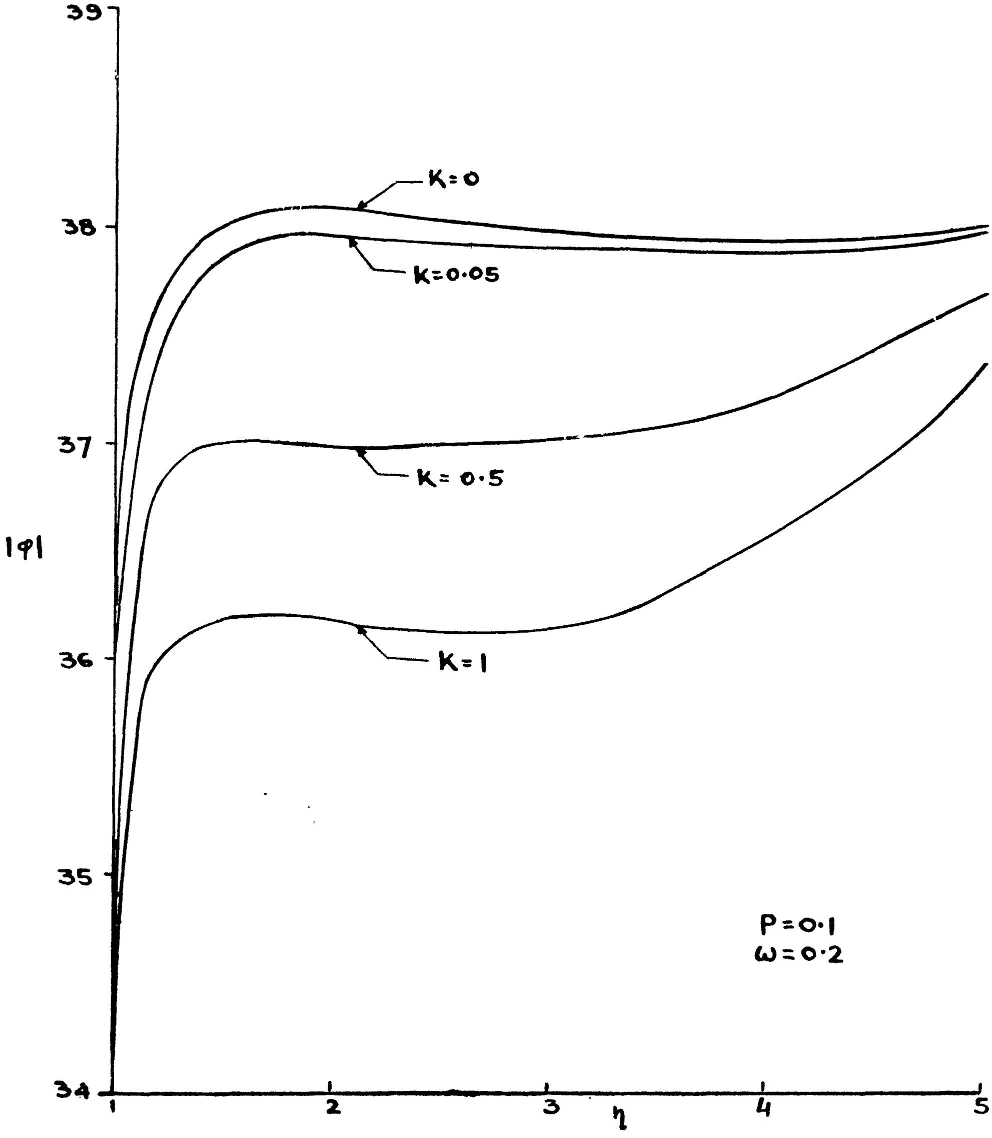

FIG. 3. $|\phi|$ in degrees vs η for different values of k.

in the range $0.5 \leqslant \eta \leqslant 2$. Values of $M = |M_r + iM_i|$ are given in Table VI, and presented graphically in Fig. 4.

In Table VII, values of M_r, M_i, ϕ computed for increasing values of ω are given. It is seen that as ω increases M_r decreases and M_i, ϕ increase with increasing ω for small values of P.

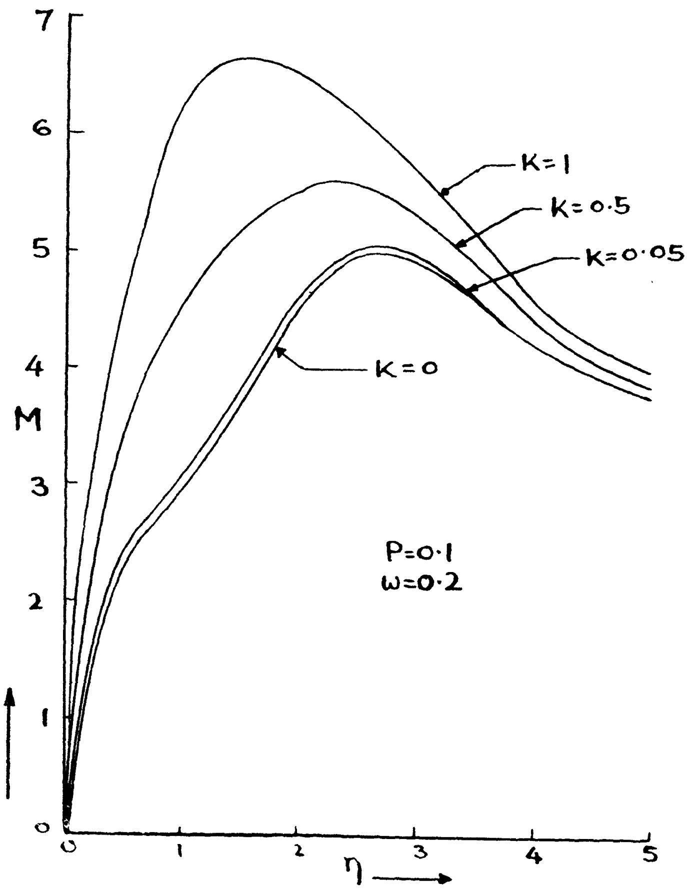

FIG. 4. $| M_r + iM_i |$ vs η for different k.

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