A GRAVITATIONALLY NON-DEGENERATE BIANCHI TYPE I COSMOLOGICAL MODEL IN GENERAL RELATIVITY

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A gravitationally non-degenerate cosmological model for a cylindrically symmetric space-time with two degrees of freedom has been derived. Various physical and geometrical properties of the model have been discussed.

Introduction

Recently, Ellis and MacCallum (1969) studied a class of homogeneous models representing perfect fluid distributions. A survey of geometrical techniques, specially homogeneous cosmological models and the Bianchi classification scheme was done by MacCallum (1973). General solutions for such a class of space times representing incoherent matter have been obtained by Heckmann and Schucking (1962). In this paper, we construct a gravitationally non-degenerate cylindrically symmetric cosmological model which is of Bianchi type I. The energy momentum tensor has been assumed to be that of a perfect fluid. Explicit expressions for pressure, density, rotation, shear and expansion have been obtained. An expression for the generalized Doppler effect in the model has also been obtained.

We consider the cylindrically symmetric metric with two degrees of freedom in the form

$$ds^2 = A^2(dx^2 - dt^2) + C^2\, dz^2 + (B^2 + D^2)\, dy^2 + 2CD\, dy\, dz \qquad \ldots(1)$$

where the metric potentials are functions of time alone. The energy momentum tensor for a perfect fluid distribution is given by

$$T_i^j = (\epsilon + p)\, v_i v^j + p\delta_i^j \qquad \ldots(2)$$

together with

$$g_{ij} v^i v^j = -1 \qquad \ldots(3)$$

p being the pressure; ϵ, the density; and v^i, the flow vector satisfying (3). We assume the co-ordinates to be co-moving, so that

$$v^1 = v^2 = v^3 = 0 \text{ and } v^4 = \frac{1}{A}\cdot$$

The field equations

$$-8\pi T_i^j = R_i^j - \tfrac{1}{2} R g_i^j + \Lambda\, \delta_i^j \qquad \ldots(4)$$

for the line element (1) are

$$8\pi p = \frac{1}{A^2}\left\{\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{B_4 C_4}{BC} - \frac{B_{44}}{B} - \frac{C_{44}}{C}\right\} - \tfrac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda \qquad \ldots(5)$$

$$8\pi p = -\frac{1}{A^2}\left\{\frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{A_4^2}{A^2}\right\} - \tfrac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_{44} + \tfrac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B} - 3\frac{C_4}{C}\right) - \tfrac{3}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda \qquad \ldots(6)$$

$$8\pi p = -\frac{1}{A^2}\left\{\frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{A_4^2}{A^2}\right\} + \tfrac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_{44} - \tfrac{1}{2}\frac{DC}{A^2B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B} - 3\frac{C_4}{C}\right) + \tfrac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 - \Lambda \qquad \ldots(7)$$

$$8\pi\epsilon = \frac{1}{A^2}\left\{\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} + \frac{B_4 C_4}{BC}\right\} - \tfrac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 + \Lambda \qquad \ldots(8)$$

and

$$\left(\frac{D}{C}\right)_{44} - \left(\frac{D}{C}\right)_4\left\{\frac{B_4}{B} - 3\frac{C_4}{C}\right\} = 0. \qquad \ldots(9)$$

Choosing the tetrad $\lambda_{i/}^{j}$ as

$$\lambda_{i/}^{j} = \begin{bmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{C} & 0 & 0 \\ 0 & -\frac{D}{BC} & \frac{1}{B} & 0 \\ 0 & 0 & 0 & \frac{1}{A} \end{bmatrix} \qquad \ldots(10)$$

the physical components of conformal curvature tensor are

$$C_{(1212)} = -C_{(3434)}$$
$$= \frac{1}{6A^2}\left[\frac{A_{44}}{A} - \frac{A_4^2}{A^2} + \frac{C_{44}}{C} - \frac{2B_{44}}{B} + \frac{B_4C_4}{BC} + \frac{3A_4B_4}{AB} - \frac{3A_4C_4}{AC} - \frac{2C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2\right] \quad ...(11)$$

$$C_{(2424)} = -C_{(1313)}$$
$$= -\frac{1}{6A^2}\left[\frac{A_{44}}{A} - \frac{A_4^2}{A^2} + \frac{B_{44}}{B} - \frac{2C_{44}}{C} + \frac{B_4C_4}{BC} + \frac{3A_4C_4}{AC} - \frac{3A_4B_4}{AB} + \frac{C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2\right] \quad ...(12)$$

$$C_{(1414)} = -C_{(2323)}$$
$$= -\frac{1}{6A^2}\left[\frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{2A_{44}}{A} + \frac{2A_4^2}{A^2} - \frac{2B_4C_4}{BC} + \frac{C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2\right] \quad ...(13)$$

$$C_{(2434)} = C_{(1213)}$$
$$= \frac{1}{4A^2}\cdot\frac{C}{B}\left[\left(\frac{D}{C}\right)_{44} + \left(\frac{D}{C}\right)_4\left\{\frac{3C_4}{C} - \frac{2A_4}{A} - \frac{B_4}{B}\right\}\right] \quad ...(14)$$

Suffix 4 indicates ordinary differentiation with respect to t.

Solution of the Field Equations

Equations. (5) – (9) are five equations in six unknowns, A, B, C, D, ϵ and p. For the complete determination of this set, we need an extra condition. We assume $C_{(1414)} = 0$. The gravitational field will obviously be of non-degenerate Petrov type I. Therefore, from eqn. (13), we have

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} - 2\left(\frac{A_4}{A}\right)_4 - \frac{2B_4C_4}{BC} + \frac{C^2}{B^2}\left\{\left(\frac{D}{C}\right)_4\right\}^2 = 0. \quad ...(15)$$

From eqn. (9), we have

$$\left(\frac{D}{C}\right)_4 = K\frac{B}{C^3} \quad ...(16)$$

where K is a constant of integration. From eqns. (5), (6), (7) and (16), we have

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} = \frac{K^2}{C^4} \quad ...(17)$$

and

$$\frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{B_4C_4}{BC} - \frac{B_{44}}{B} + \left(\frac{A_4}{A}\right)_4 = \frac{K^2}{2C^4}. \quad ...(18)$$

From eqns. (15), (16) and (17), we have

$$\left(\frac{A_4}{A}\right)_4 = \frac{C_{44}}{C} - \frac{B_4 C_4}{BC}. \quad \text{...(19)}$$

Using (17) and (19) in (18), we have

$$2\,\frac{A_4}{A}\,\frac{(BC)_4}{BC} = \frac{4B_4C_4}{BC} - \frac{C_{44}}{C} + \frac{B_{44}}{B}. \quad \text{...(20)}$$

Also, from eqn. (18), we have

$$\frac{A_4}{A} = \frac{(BC)_4}{2BC} + \frac{L}{BC} \quad \text{...(21)}$$

where L is a constant of integration. From eqn. (17), we have

$$\left(\frac{B_4}{B}\right)_4 + \frac{B_4^2}{B^2} = \frac{C_{44}}{C} - \frac{K^2}{C^4}. \quad \text{...(22)}$$

Using (17) and (21) in (20), we have

$$\left(\frac{B_4}{B} - \frac{C_4}{C}\right)^2 + 2L\frac{(BC)_4}{B^2C^2} = -\frac{K^2}{C^4}. \quad \text{...(23)}$$

Putting $B = e^{(U+V)/2}$, $C = e^{(U-V)/2}$ in (22) and (23), we have

$$V_{44} + U_4 V_4 = -K^2\, e^{2V-2U} \quad \text{...(24)}$$

and

$$V_4^2 + 2Lu_4\, e^{-U} = -K^2 e^{2V-2U}. \quad \text{...(25)}$$

Equation (24) on integration gives

$$e^U V_4 = \sqrt{M - K^2\, e^{2V}} \quad \text{...(26)}$$

M being a constant of integration. From (25) and (26), we have

$$e^U = \frac{N - Mt}{2L} \quad \text{...(27)}$$

where N is a constant of integration. Equation (26) leads to

$$e^{-V} = \frac{K}{\sqrt{M}} \cosh\left\{\frac{2L}{\sqrt{M}} \log a\,(N - Mt)\right\} \quad \text{...(28)}$$

a being a constant of integration. Hence,

$$B^2 = \frac{\sqrt{M}\,(N - Mt)}{2LK}\,\text{sech}\left\{\frac{2L}{\sqrt{M}} \log a\,(N - Mt)\right\} \quad \text{...(29)}$$

and

$$C^2 = \frac{K(N - Mt)}{2L\sqrt{M}} \cosh\left\{\frac{2L}{\sqrt{M}} \log a\,(N - Mt)\right\}. \qquad \text{...(30)}$$

Using (29) and (30) in (16), we have

$$\left(\frac{D}{C}\right)_4 = \frac{2LM \operatorname{sech}^2\left\{\frac{2L}{\sqrt{M}} \log a\,(N - Mt)\right\}}{K(N - Mt)} \qquad \text{...(31)}$$

which on integration gives

$$D = C\left[k - \frac{\sqrt{M}}{K} \tanh\left\{\frac{2L}{\sqrt{M}} \log a\,(N - Mt)\right\}\right] \qquad \text{...(32)}$$

where k is a constant of integration. From (21), (29) and (30), we have

$$A = \{b(N - Mt)\}^{(M-4L^2)/2M} \qquad \text{...(33)}$$

b being a constant of integration. After suitable transformations of co-ordinates, the metric reduces to the form

$$\begin{aligned} ds^2 = {} & \frac{\{bT\}^{(\beta^2-4L^2)/\beta^2}}{\beta^4} (dX^2 - dT^2) + T \cosh\phi\, dZ^2 \\ & + [(\alpha^2 + \beta^2)\cosh\phi - 2\alpha\beta \sinh\phi]\, T dY^2 \\ & + 2(\alpha\cosh\phi - \beta\sinh\phi)\, T dY\, dZ \end{aligned} \qquad \text{...(34)}$$

α and β being arbitrary constants of integration and

$$\phi = \frac{2L}{\beta} \log a\, T.$$

Some Physical and Geometrical Features

The pressure and density for the model (34) are given by

$$8\pi p = \frac{3}{\beta^2(bT)^{1-(4L^2/\beta^2)}}\left[\frac{\beta^2}{4T^2} - \frac{L^2}{T^2}\right] - \Lambda \qquad \text{...(35)}$$

$$8\pi\epsilon = \frac{3}{\beta^2(bT)^{1-(4L^2/\beta^2)}}\left[\frac{\beta^2}{4T^2} - \frac{L^2}{T^2}\right] + \Lambda. \qquad \text{...(36)}$$

The scalar of expansion θ calculated for the flow vector v^i is given by

$$\theta = \frac{3\beta^2 - 4L^2}{2\beta^2 T(bT)^{(\beta^2-4L^2)/2\beta^2}}. \qquad \text{...(37)}$$

The rotation ω is identically zero and the non-vanishing components of the shear tensor σ_{ij} are given by

$$\sigma_{11} = -\frac{4L^2}{3T\beta^6} \cdot (bT)^{(\beta^2-4L^2)/2\beta^2} \qquad \ldots(38)$$

$$\sigma_{22} = \frac{\cosh\phi}{3\beta^2(bT)^{(\beta^2-4L^2)/2\beta^2}}\,[2L^2 + 3\alpha\beta\tanh\phi] \qquad \ldots(39)$$

$$\sigma_{33} = -\frac{\cosh\phi}{2\beta^2(bT)^{(\beta^2-4L^2)/2\beta^2}}\left[\left\{\frac{8L^2\alpha\beta}{3} - 2L\beta(\alpha^2+\beta^2)\right\}\tanh\phi + \frac{4L}{3}\{3\alpha\beta^2 - L(\alpha^2+\beta^2)\}\right] \qquad \ldots(40)$$

and

$$\sigma_{23} = -\frac{L\cosh\phi}{\beta(bT)^{(\beta^2-4L^2)/2\beta^2}}\left[\left(\frac{2L}{3}-\alpha\right)\tanh\phi + \left(\beta - \frac{2L\alpha}{3\beta}\right)\right]. \qquad \ldots(41)$$

The model has to satisfy the reality conditions (Ellis 1971)

(i) $(\epsilon + p) > 0$

and

(ii) $(\epsilon + 3p) > 0$.

Condition (i) leads to $\beta^2 > 4L^2$.

Condition (ii) is then obviously satisfied, if $\Lambda < 0$. Since $T = 0$ is a singularity, we interpret this result that the model expands from a singular state at $T = 0$ and goes on expanding indefinitely. However, if $\Lambda > 0$, it requires that

$$T^{3-(4L^2/\beta^2)} < \frac{3(\beta^2 - 4L^2)}{2\Lambda\beta^2(b)^{(\beta^2-4L^2)/\beta^2}}. \qquad \ldots(42)$$

In this case, the model expands from its singular state at time $T = 0$ and exists during the period given by (42).

The red shift in the model (34) is given by

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{T_2^{2L^2/\beta^2}\left\{(bT)^{(\beta^2-4L^2)/\beta^2} - U^2\beta^4\right\}^{1/2}\left\{\gamma\,\mathrm{sech}^{1/2}\phi + U_z\,T_1^{2L^2/\beta^2}\right\}}{\gamma T_1^{2L^2/\beta^2}\,(bT)^{(\beta^2-4L^2)/2\beta^2}}$$

where u is the velocity of the source at the time of emission and U_z is the z-component of the velocity. The non-vanishing components of conformal curvature tensor are given by

$$C_{(1212)} = -C_{(1313)} = \frac{L(4L^2 - \beta^2)\tanh\phi}{2\beta^3 T^2 (bT)^{(\beta^2-4L^2)/\beta^2}} \qquad \ldots(43)$$

and

$$C_{(2434)} = \frac{L \operatorname{sech}\phi(\beta^2 - 4L^2 - 8L\beta\tanh\phi)}{2\beta^3 T^2 (bT)^{(\beta^2-4L^2)/\beta^2}}. \qquad \ldots(44)$$

Hence, the flow vector is geodetic and the model represents an expanding, shearing but non-rotating universe.

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