THERMAL STRESSES IN A SOLID ELASTIC CONE*

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This paper deals with a formal solution of the thermoelastic problem of an elastic circular cone, due to a nucleus of strain situated on the axis of the cone at a finite distance from the apex. The Mellin transform technique is employed together with Boussinesq-papkovitch potentials to obtain the complex integral representations of the temperature stresses required. The results achieved are compared with those obtained earlier.

1.1. INTRODUCTION

The stress analysis of circular cones and of solids of revolution bounded by one or two cones, etc. has been the object of numerous investigations in the classical theories of elasticity and thermoelasticity. Solutions for a semi-infinite solid cone under a concentrated force of arbitrary orientation applied at the vertex were established by Michell (1900). Föppl (1921) studied the analogous pure torsion problem. Michell's results were extended by Neuber (1934) to the case of a conical shell of varying thickness. Knops (1958) rederived the results of Neuber (1934), in a different manner. Muki and Sternberg (1960) obtained solutions for the steady state thermal stresses in an elastic cone subjected to a discontinuous surface temperature field using the modified Boussinesq-papkovitch's general method to account for the presence of temperature field, which was introduced by McDowell and Sternberg (1957).

In this paper, we seek solutions for the thermal stresses in a solid elastic circular cone free from external loads, due to a nucleus of thermoelastic strain situated on the axis of the cone at a finite distance from the apex. The Mellin transform technique is employed together with Boussinesq-papkovitch's scalar potential functions to obtain the complex integral representations of the temperature stresses required. Fourier-Legendre expansions of these scalar potential functions and that of the Dirac delta function are made use of to obtain the particular solution of the thermoelastic problem.

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The problem is treated on the basis of the classical linear theory of thermoelasticity. Throughout the analysis, the material is supposed to be homogeneous and isotropic with respect to both its thermal and mechanical responses and all physical properties are regarded to be independent of temperature. The results obtained are compared with those reported earlier.

1.2. Basic Equations and Formulation of the Problem

Consider a medium occupying an isotropic homogeneous region of space D with boundary B. The spherical polar coordinates (r, θ, ϕ) are introduced by means of the mapping :

$$\left.\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}\right\} \qquad \ldots(1.2.1)$$

$$0 \leqslant r < \infty, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi \leqslant 2\pi.$$

Suppose the axis of the cone in question coincides with the z-axis, with its apex at the origin, and let θ_0 $(0 < \theta_0 \leqslant \pi/2)$ be its semi-vertical angle.

For convenience, let us introduce the following auxiliary variables :

$$p = \cos\theta, \quad q = \sin\theta$$

$$p_0 = \cos\theta_0, \quad q_0 = \sin\theta_0. \qquad \ldots(1.2.2)$$

The region D under consideration is then given by

$$0 < r < \infty, \quad p_0 < p \leqslant 1, \quad (0 \leqslant p_0 < 1),$$

while the boundary B corresponds to $p = p_0$.

Let the nucleus of thermoelastic strain be situated at the point $(0, 0, z')$ on the axis of the cone at a finite distance z' from the apex. Then the temperature distribution is given as

$$T = \delta(x)\,\delta(y)\,\delta(z - z') \qquad \ldots(1.2.3)$$

where $\delta = \delta(x)$ is the Dirac delta function.

The displacements and stresses due to the action of this nucleus are given in terms of the thermo-displacement potential F satisfying the equation (Nowacki 1962)

$$\nabla^2 F = mT \qquad \ldots(1.2.4)$$

where ∇^2 is the Laplacian operator and $m = \dfrac{(1 + \upsilon)\alpha_t}{(1 - \upsilon)}$.

For the removal of the stress 'residuals' on the surface of the cone produced by the nucleus, the Boussinesq-papkovitch approach is used, according to which the general solution of the displacement equations of equilibrium in the case of torsion-free rotational symmetry and absence of body forces admits the representation as:

$$U_i = \phi_{,i} - (3 - 4\upsilon)\,\psi_i + x_j\psi_{j,i} \quad (i = j = 1, 2.\ 3) \qquad \ldots(1.2.5)$$

$$\phi_{,jj} = 0, \quad \psi_{i,jj} = 0 \qquad \ldots(1.2.6)$$

where ϕ and ψ are Boussinesq-papkovitch potential functions and υ is the Poisson's ratio and the subscript comma denotes partial differentiation.

In the case of axisymmetry about the z-axis, the general solutions (1.2.5) and (1.2 6) of the isothermal elastostatic field equations remain complete, if ϕ and ψ_i are restricted by

$$\phi = \phi(r, p), \quad \psi_3 = \psi(r, p)$$

$$\psi_1 = \psi_2 = 0. \qquad \ldots(1.2.7)$$

Whence (1.2.6) now reduces to

$$\nabla^2\phi(r, p) = 0, \quad \nabla^2\psi(r, p) = 0 \qquad \ldots(1.2.8)$$

in which the Laplacian operator ∇^2 is given as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial p}\left[(1 - p^2)\frac{\partial}{\partial p}\right]. \qquad \ldots(1.2.9)$$

In spherical polar coordinates, eqn. (1.2.5) becomes

$$\bar{\bar{U}}_r = \frac{\partial\phi}{\partial r} + pr\frac{\partial\psi}{\partial r} - (3 - 4\upsilon)\,p\psi$$

$$\bar{\bar{U}}_\theta = -\frac{q}{r}\frac{\partial\phi}{\partial p} - pq\frac{\partial\psi}{\partial p} + (3 - 4\upsilon)\,q\psi; \quad \bar{\bar{U}}_\phi = 0. \qquad \ldots(1.2.10)$$

The corresponding stress field in terms of the stress functions $\phi(r, p)$ and $\psi(r, p)$ is given as

$$\frac{\bar{\bar{\sigma}}_{rr}}{2\mu} = \frac{\partial^2\phi}{\partial r^2} + rp\frac{\partial^2\psi}{\partial r^2} - 2(1 - \upsilon)\,p\frac{\partial\psi}{\partial r} - \frac{2\upsilon q^2}{r}\frac{\partial\psi}{\partial p}$$

$$\frac{\bar{\bar{\sigma}}_{\theta\theta}}{2\mu} = \frac{q^2}{r^2}\frac{\partial^2\phi}{\partial p^2} - \frac{p}{r^2}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{pq^2}{r}\frac{\partial^2\psi}{\partial p^2}$$

$$+ (1 - 2\upsilon)\,p\frac{\partial\psi}{\partial r} - \frac{1}{r}[1 + (1 - 2\upsilon)\,q^2]\frac{\partial\psi}{\partial p}$$

(*equation continued on p.* 259)

$$\frac{\bar{\bar{\sigma}}_{\phi\phi}}{2\mu} = \frac{-p}{r^2}\frac{\partial\phi}{\partial p} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial p} + (1-2\upsilon)\left[p\frac{\partial\psi}{\partial r} + \frac{q^2}{r}\frac{\partial\psi}{\partial p}\right]$$

$$\frac{\bar{\bar{\sigma}}_{r\theta}}{2\mu} = q\left[\frac{1}{r^2}\frac{\partial\phi}{\partial p} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial p} - p\frac{\partial^2\psi}{\partial r\partial p} + (1-2\upsilon)\frac{\partial\psi}{\partial r} + 2(1-\upsilon)\frac{p}{r}\frac{\partial\psi}{\partial p}\right]$$

$$\bar{\bar{\sigma}}_{r\phi} = 0, \quad \bar{\bar{\sigma}}_{\theta\phi} = 0. \qquad \ldots(1.2.11)$$

Similarly, eqn. (1.2.4) in spherical coordinates reduces to

$$\nabla^2 F = \frac{m}{2\pi}\frac{\delta(r-r')\,\delta(\theta)}{r^2\sin\theta} \qquad \ldots(1.2.12)$$

The displacement and the stress fields of the thermoelastic displacement potential in the spherical polar coordinates are expressed as :

$$\bar{U}_r = \frac{\partial F}{\partial r}, \quad \bar{U}_\theta = -\frac{q}{r}\frac{\partial F}{\partial p}, \quad \bar{U}_\phi = 0 \qquad \ldots(1.2.13)$$

and

$$\frac{\bar{\sigma}_{rr}}{2\mu} = \frac{\partial^2 F}{\partial r^2}$$

$$\frac{\bar{\sigma}_{\theta\theta}}{2\mu} = -\frac{q^2}{r^2}\frac{\partial^2 F}{\partial p^2} - \frac{p}{r^2}\frac{\partial F}{\partial p} + \frac{1}{r}\frac{\partial F}{\partial r}$$

$$\frac{\bar{\sigma}_{\phi\phi}}{2\mu} = \frac{p}{r^2}\frac{\partial F}{\partial p} + \frac{1}{r}\frac{\partial F}{\partial r}$$

$$\frac{\bar{\sigma}_{r\theta}}{2\mu} = q\left[\frac{1}{r^2}\frac{\partial F}{\partial p} - \frac{1}{r}\frac{\partial^2 F}{\partial r\partial p}\right]$$

$$\bar{\sigma}_{r\phi} = 0, \quad \bar{\sigma}_{\theta\phi} = 0. \qquad \ldots(1.2.14)$$

Thus, the final displacement and stress fields in the spherical polar coordinates are expressed as follows :

$$U_r = \frac{\partial F}{\partial r} + \frac{\partial\phi}{\partial r} + pr\frac{\partial\psi}{\partial r} - (3-4\upsilon)\,p\psi$$

$$U_\theta = \frac{-q}{r}\frac{\partial F}{\partial p} - \frac{q}{r}\frac{\partial\phi}{\partial p} - pq\frac{\partial\psi}{\partial p} + (3-4\upsilon)\,q\psi$$

$$U_\phi = 0 \qquad \ldots(1.2.15)$$

and

$$\frac{\sigma_{rr}}{2\mu} = \frac{\partial^2 F}{\partial r^2} + \frac{\partial^2 \phi}{\partial r^2} + rp\frac{\partial^2 \psi}{\partial r^2} - 2(1-\upsilon)\, p\frac{\partial \psi}{\partial r} - \frac{2\upsilon q^2}{r}\frac{\partial \psi}{\partial p}$$

$$\frac{\sigma_{\theta\theta}}{2\mu} = \frac{q^2}{r^2}\frac{\partial^2 F}{\partial p^2} - \frac{p}{r^2}\frac{\partial F}{\partial p} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{q^2}{r^2}\frac{\partial^2 \phi}{\partial p^2} - \frac{p}{r^2}\frac{\partial \phi}{\partial p} + \frac{1}{r}\frac{\partial \phi}{\partial r}$$
$$+ \frac{pq^2}{r}\frac{\partial^2 \psi}{\partial p^2} + (1-2\upsilon)\, p\frac{\partial \psi}{\partial r} - [1 + (1-2\upsilon)\, q^2]\frac{1}{r}\frac{\partial \psi}{\partial p}$$

$$\frac{\sigma_{\phi\phi}}{2\mu} = \frac{-p}{r^2}\frac{\partial F}{\partial p} + \frac{1}{r}\frac{\partial F}{\partial r} - \frac{p}{r^2}\frac{\partial \phi}{\partial p} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial p}$$
$$+ (1-2\upsilon)\left[p\frac{\partial \psi}{\partial r} + \frac{q^2}{r}\frac{\partial \psi}{\partial p}\right]$$

$$\frac{\sigma_{r\theta}}{2\mu} = q\left[\frac{1}{r^2}\frac{\partial F}{\partial p} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial p} + \frac{1}{r^2}\frac{\partial \phi}{\partial p} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial p} - p\frac{\partial^2 \psi}{\partial r \partial p}\right.$$
$$\left. + (1-2\upsilon)\frac{\partial \psi}{\partial r} + 2(1-\upsilon)\,\frac{p}{r}\frac{\partial \psi}{\partial p}\right]$$

$$\sigma_{r\phi} = \sigma_{\theta\phi} = 0. \qquad \text{...(1.2.16)}$$

The boundary conditions to be satisfied on the surface of the cone (on $p = p_0$) are

$$\sigma_{\theta\theta}(r, p_0) = 0, \quad \sigma_{r\theta}(r, p_0) = 0 \quad (0 < r < \infty). \qquad \text{...(1.2.17)}$$

To these boundary conditions we adjoin the regularity conditions

$$\sigma_{rr},\ \sigma_{\theta\theta},\ \sigma_{\phi\phi},\ \sigma_{r\theta} \to 0 \quad \text{as} \quad r \to \infty \qquad \text{...(1.2.18)}$$

as well as the requirement that the entire stress distribution be bounded in $D + B$ for $0 \leqslant r < \infty$, $p_0 \leqslant p \leqslant 1$.

The problem under consideration thus reduces to the determination of the solutions F, ϕ and ψ of (1.2.12) and (1.2.8), such that the stresses (1.2.16) obey (1.2.17) and (1.2.18).

1.3. Application of the Mellin Transform

Let $\hat{F}(p, s)$, $\hat{\phi}(p, s)$ and $\hat{\psi}(p, s)$ be the Mellin transforms with respect to r of $r^{-2}F(r, p)$, $r^{-2}\phi(r, p)$ and $r^{-1}\psi(r, p)$, s being the transform parameter (Doetsch 1950).

Thus,

$$\hat{F}(p, s) = \int_0^\infty F(r, p)\, r^{s-3}\, dr$$

(equation continued on p. 261)

$$\hat{\phi}(p, s) = \int_0^\infty \phi(r, p)\, r^{s-3}\, dr$$

$$\hat{\psi}(p, s) = \int_0^\infty \psi(r, p)\, r^{s-2}\, dr \qquad \text{...(1.3.1)}$$

Further, following Muki and Sternberg (1960), we assume that

$$r^{-2}F,\ r^{-1}\frac{\partial F}{\partial r},\ r^{-2}\phi,\ r^{-1}\frac{\partial \phi}{\partial r},\ r^{-1}\psi, \frac{\partial \psi}{\partial r} = O(r^{-s^*}) \qquad \text{...(1.3.2)}$$

as $r \to \infty$ $(s^* > 0)$.

In addition, assume that the six functions appearing in (1.3.2) remain bounded as $r \to 0$. Then (1.3.1) and integration by parts imply the identities of the type

$$\int_0^\infty \frac{\partial F}{\partial r}\, r^{s-2}\, dr = -(s-2)\,\hat{F};$$

$$\int_0^\infty \frac{\partial^2 F}{\partial r^2}\, r^{s-1}\, dr = (s-1)(s-2)\,\hat{F}; \text{ ... etc.} \qquad \text{...(1.3.3)}$$

provided s is confined to the strip $0 < \text{Re}(s) \leqslant s^*$ of the complex s-plane.

Applying the Mellin transform to (1.2.12) with the help of (1.2.9), (1.3.1) and (1.3.3), we get

$$\frac{d}{dp}\left[(1-p^2)\frac{\partial \hat{F}}{\partial p}\right] + (s-3)(s-2)\,\hat{F}$$

$$= \frac{m}{2\pi}\,\frac{\delta(\theta)}{\sin\theta}\int_0^\infty \delta(r-r')\, r^{s-3}\, dr$$

$$= \frac{m}{2\pi}\,\frac{\delta(\theta)}{\sin\theta}\,(r')^{s-3}. \qquad \text{...(1.3.4)}$$

We have made use of the equation

$$\int_0^\infty \delta(t-a)\, G(t)\, dt = G(a). \qquad \text{...(1.3.5)}$$

To this end, we expand the odd function $\delta(\theta)$ cosec θ into a series of Legendre polynomials (Kishan Rao and Srinivas Rao 1973),

$$\frac{\delta(\theta)}{\sin\theta} = \sum_{s=0}^{\infty} a_s P_{2s+1}(p) \qquad 0 \leqslant p \leqslant 1$$

where

$$a_s = (4s+3)\int_0^1 \frac{\delta(\theta)}{\sin\theta}\, P_{2s+1}(\cos\theta)\sin\theta\, d\theta$$

$$= (4s+3)\; P_{2s+1}(1) = (4s+3)$$

Hence,

$$\frac{\delta(\theta)}{\sin\theta} = \sum_{s=0}^{\infty} (4s+3)\, P_{2s+1}(p). \qquad \text{...(1.3.6)}$$

Let us suppose that (1.3.4) admits solution in the form

$$\hat{F} = R(s)\, P_{2s+1}(p). \qquad \text{...(1.3.7)}$$

Then substituting (1.3.7) and (1.3.6) in (1.3.4), we get

$$R(s) = -\frac{m}{2\pi}\,\frac{(4s+3)\,(r')^{s-3}}{3(s+4)\,(3s-1)}. \qquad \text{...(1.3.8)}$$

Therefore,

$$\hat{F} = \left[\frac{-\,m(4s+3)\,(r')^{s-3}}{6\pi(s+4)\,(3s-1)}\right] P_{2s+1}(p). \qquad \text{...(1.3.9)}$$

Similarly, applying the Mellin transform to eqn. (1.2.8), after multiplying the second of (1.2.8) by r, with the help of (1.2.9), (1.3.1) and (1.3.3), we get

$$\frac{d}{dp}\left[(1-p^2)\frac{d\hat{\phi}}{dp}\right] + (s-3)\,(s-2)\,\hat{\phi} = 0$$

$$\frac{d}{dp}\left[(1-p^2)\frac{d\hat{\psi}}{dp}\right] + (s-2)\,(s-1)\,\hat{\psi} = 0. \qquad \text{...(1.3.10)}$$

Next, let $\hat{U}_r(p, s)$, $\hat{U}_\theta(p, s)$ be the Mellin transform of $r^{-1}U_r(r, p)$, $r^{-1}U_\theta(r, p)$ and denote by $\hat{\sigma}_{rr}(p, s)$, $\hat{\sigma}_{\theta\theta}(p, s)$, etc. the transforms of the corresponding components of stress. Explicitly,

$$\hat{U}_r(p, s) = \int_0^\infty U_r(r, p)\, r^{s-2}\, dr \;\dots\; \text{etc.}$$

$$\hat{\sigma}_{rr}(p, s) = \int_0^\infty \sigma_{rr}(r, p)\, r^{s-2}\, dr \;\dots\; \text{etc.} \qquad \text{...(1.3.11)}$$

Then eqns. (1.2.15) and (1.2.16) with the help of (1.3.1), (1.3.3), (1.3.7), (1.3.10) and (1.3.11) reduce to

$$\hat{U}_r = -(s-2)\hat{F} - (s-2)\hat{\phi} - p[s + 2(1-2\upsilon)]\hat{\psi}$$

$$\hat{U}_\theta = -q\left[\frac{d\hat{F}}{dp} + \frac{d\hat{\phi}}{dp} + p\frac{d\hat{\psi}}{dp} - (3-4\upsilon)\hat{\psi}\right]. \qquad \ldots(1.3.12)$$

and

$$\frac{\hat{\sigma}_{rr}}{2\mu} = (s-2)(s-1)\hat{F} + (s-1)(s-2)\hat{\phi}$$
$$+ (s-1)(s+2-2\upsilon)p\hat{\psi} - 2\upsilon q^2\frac{d\hat{\psi}}{dp}$$

$$\frac{\hat{\sigma}_{\theta\theta}}{2\mu} = -s(4s+7)\hat{F} + p\frac{d\hat{F}}{dp} - (s-2)^2\hat{\phi} + p\frac{d\hat{\phi}}{dp}$$
$$-(s-1)(s-1-2\upsilon)p\hat{\psi} + [1-(3-2\upsilon)q^2]\frac{d\hat{\psi}}{dp}$$

$$\frac{\hat{\sigma}_{\phi\phi}}{2\mu} = -(s-2)\hat{F} - p\frac{d\hat{F}}{dp} - (s-2)\hat{\phi} - p\frac{d\hat{\phi}}{dp} - (1-2\upsilon)(s-1)p\hat{\psi}$$
$$+ [(1-2\upsilon)q^2 - 1]\frac{d\hat{\psi}}{dp}$$

$$\frac{\hat{\sigma}_{r\theta}}{2\mu} = q\left[(s-1)\frac{d\hat{F}}{dp} + (s-1)\frac{d\hat{\phi}}{dp} - (1-2\upsilon)(s-1)\hat{\psi}\right.$$
$$\left. + (s+1-2\upsilon)p\frac{d\hat{\psi}}{dp}\right]. \qquad \ldots(1.3.13)$$

Finally, the boundary conditions (1.2.17) in the transformed domain become

$$\hat{\sigma}_{\theta\theta}(p, s) = 0, \quad \hat{\sigma}_{r\theta}(p, s) = 0 \qquad \ldots(1.3.14)$$

Taking into account (1.3.7) and keeping in mind that the thermal displacements and stresses sought must be regular along the axis of the cone $\theta = 0$, i.e., for $p = 1$, we take the solution of (1.3.10) in the form

$$\hat{\phi}(p, s) = A(s)\,P_{s-3}(p)$$

$$\hat{\psi}(p, s) = B(s)\,P_{s-2}(p). \qquad \ldots(1.3.15)$$

The arbitrary functions $A(s)$ and $B(s)$ appearing in (1.3.15) are to be determined from the transformed boundary conditions (1.3.14). Substituting (1.3.15) into the second and the last of (1.3.13) and invoking (1.3.14) subsequently, we get a pair of simultaneous equations involving $A(s)$ and $B(s)$. These equations can further be simplified with the aid of the recursion relations,

$$q^2 P'_s(p) = s\,[P_{s-1}(p) - pP_s(p)]$$

$$(2s+1)\,pP_s(p) = (s+1)\,P_{s+1}(p) + sP_{s-1}(p) \qquad \ldots(1.3.16)$$

where prime indicates differentiation with respect to the argument.

Setting

$$\begin{aligned}\triangle(p_0, s) &= (s-1)^2 (s-2)^2 p_0 [P_{s-2}(p_0)]^2 \\ &\quad + (s-1)(s-2)\, q_0^2\, P_{s-2}(p_0)\, P'_{s-2}(p_0) \\ &\quad + p_0 [2(\upsilon - 1) + (s-1)(s-2)\, q_0^2]\, [P'_{s-2}(p_0)]^2 \qquad \ldots(1.3.17)\end{aligned}$$

and making repeated use of (1.3.16), we write the final formulae for $\hat{\phi}$, $\hat{\psi}$ into the form

$$\hat{\phi} = \frac{RL_1}{\triangle} P_{s-3}(s)$$

$$\hat{\psi} = \frac{RL_2}{\triangle} P_{s-2}(s) \qquad \ldots(1.3.18)$$

where

$$\begin{aligned}L_1 &= P'_{2s+1}(p_0)\, [P'_{s-2}(p_0)\, \{2(1-\upsilon) < 1 + (s-2)\, q_0^2 >\} \\ &\quad + (s-1)(s-2)(s-2\upsilon)\, p_0 P_{s-2}(p_0)] \\ &\quad - s(4s+7)\, P_{2s+1}(p_0)\, [(s+1-2\upsilon)\, p_0 P'_{s-2}(p_0) \\ &\quad - (s-1)(1-2\upsilon)\, P_{s-2}(p_0)]\end{aligned}$$

and

$$\begin{aligned}L_2 &= (s-1)\, [s(4s+7)\, P_{2s+1}(p_0)\, P'_{s-3}(p_0) \\ &\quad - (s-2)^2 P'_{2s+1}(p_0)\, P_{s-3}(p_0)]. \qquad \ldots(1.3.19)\end{aligned}$$

Inserting (1.3.18) and (1.3.19) into (1.3.13) and using (1.3.16), we get the expressions for the transformed stress components in the form :

$$\begin{aligned}\frac{\hat{\sigma}_{rr}}{2\mu R} &= \Big[(s-1)(s-2) \Big\{ P_{2s+1}(p) + \frac{L_1}{\triangle} P_{s-1}(p) \Big\} \\ &\quad + \frac{L_2}{\triangle} \{(s-1)(s+2-2\upsilon)\, pP_{s-2}(p) - 2\upsilon q^2 P'_{s-2}(p)\} \Big]\end{aligned}$$

$$\frac{\hat{\sigma}_{\theta\theta}}{2\mu R} = \Big[- s(4s + 7)\, P_{2s+1}(p) + pP'_{2s+1}(p)$$
$$- \frac{L_1}{\Delta} \{(s-2)^2 P_{s-3}(p) - pP'_{s-3}(p)\}$$
$$- \frac{L_2}{\Delta} \{(s-1)(s-1-2\upsilon)\, pP_{s-2}(p)$$
$$+ < 1 - (3 - 2\upsilon)\, q^2 > P'_{s-2}(p)\} \Big]$$

$$\frac{\sigma_{\phi\phi}}{2\mu R} = - \Big[(s-2)\, P_{2s+1}(p) + pP'_{2s+1}(p)$$
$$+ \frac{L_1}{\Delta} \{(s-2)\, P_{s-3}(p) - pP'_{s-3}(p)\}$$
$$+ \frac{L_2}{\Delta} \{(s-1)(1-2\upsilon)\, pP_{s-2}(p)$$
$$+ < 1 - (1 - 2\upsilon)\, q^2 > P'_{s-2}(p)\} \Big]$$

$$\frac{\hat{\sigma}_{r\theta}}{2\mu R} = q \Big[(s-1)\, P'_{2s+1}(p) + (s-1) \frac{L_1}{\Delta} P'_{s-3}(p)$$
$$- \frac{L_2}{\Delta} \{(s-1)(1-2\upsilon)\, P_{s-2}(p) - (s+1-2\upsilon)\, pP'_{s-2}(p)\} \Big].$$
...(1.3.20)

For brevity, the analogous formulae for the displacement components $\hat{U}_r$, $\hat{U}_\theta$ are omitted.

In order to pass from the transformed stress functions and stresses to their antecedents in the physical domain, we have to make an appeal to the inversion Theorem for the Mellin transform.

By virtue of (1.3.1) and (1.3.11), we arrive at

$$F(r, p) = \frac{r^2}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{F}(p, s)\, r^{-s}\, ds$$

$$\phi(r, p) = \frac{r^2}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\phi}(p, s)\, r^{-s}\, ds$$

$$\psi(r, p) = \frac{r}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\psi}(p, s)\, r^{-s}\, ds \qquad ...(1.3.21)$$

and

$$\sigma_{rr}(r, p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\sigma}_{rr}(p, s)\, r^{-s}\, ds \ldots \text{etc.} \qquad \ldots(1.3.22)$$

The integrands in (1.3.21) and (1.3.22) are fully accounted for through (1.3.18), (1.3.19) and (1.3.20).

This completes the formal solution in complex integral form of the thermal stress problem under consideration.

In conclusion, we merely mention that the solution to the thermoelastic problem of cone established in this investigation when $\theta_0 = \pi/2$ is reducible to the corresponding result for the semi-infinite elastic solid obtained by Sen (1951). In particular, it is easily seen from (1.3.20) that $\theta_0 = \pi/2$, i.e., $(p_0 = 0)$ the normal stress $\hat{\sigma}_{rr}(p, s)$ vanishes identically along the axis $\theta = 0$ $(p = 1)$ and in agreement with Sen (1951).

REFERENCES

Das, S. C., and Kishan Rao, K. (1968). Axisymmetric thermal stresses in a spheroidal shell of arbitrary thickness. *Acta Tech. Draha*, No. 6, 830-43.

Doetsch, G. (1950). Handbuck der Laplace-Transformation, Vol. I. Birkhauser, Basel.

Föppl, L. (1921). Die Torsion runder Stabe von Veranderlichem Querschnitt. *Bayer. Akad. Wiss.*, 61.

Goodier, J. N. (1937). On the integration of the thermo-elastic equations. *Phil. Mag.*, **23**, Ser. 7, 1017-32.

Kishan Rao, K. (1975) Thermal stresses in bi-spherical coordinates. *Jap. J. appl. Phys.*, **14** (8), 1137-44.

——————— (1975). Stress concentration around a small spheroidal inclusion due to a nucleus of thermo-elastic strain in an infinite space. *Jap. J. appl. Phys.*, **14** (8), 1127-35.

Kishan Rao, K., and Srinivas Rao, B. N. (1973). Nucleus of thermo-elastic strain in a half space. *Proc. Fourth Iran National Math. Conf. Tehran, Iran*, 169-183.

Knops, R. J. (1958). On the variation of the Poisson's ratio in the solution of the elastic problems. *Q. Jl Mech. appl. Math.*, **11** (3), 326-50.

McDowell, E. L., and Sternberg, E. (1957). On the steady state thermo-elastic problem for the half-space. *Q. appl. Math.*, **14** (No. 4), 381-98.

Melan, E., and Parkus, H. (1953). Warmespannungen. Springer-Verlag, Berlin.

Michell, J. H. (1900). Stress analysis of the semi-infinite solid cone under a concentrated force. *Proc. Lond. math. Soc.*, **32**, 23.

Mitchell, T. P., and Weese, J. A. (1960). Stress distributions analysed in bi-spherical co-ordinates. *J appl. Mech.*, **27**(4), 726-32.

Muki, R., and Sternberg, E. (1960). Steady state thermal stresses of the solid cone. *ZAMP*, **11**, 471-96.

Neuber, H. (1934). Ein neuer Ansatz Zur Losung raumlicher probleme Elastizitats theorie. *ZAMP*, **14** (4), 203-12.

Nowacki, W. (1962). Thermo-elasticity. Addison-Wesely Pub. Co., Reading, Mass.

Sen, B. (1951). Note on the stresses produced by nucleus of thermo-elastic strain in a semi-infinite elastic solid. *Q. appl. Math.*, **8** (4), 365-69.