ON COMMON FIXED POINTS IN UNIFORM SPACES II

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A common fixed point theorem for a pair of operators, defined on a separated uniform space is proved.

1. INTRODUCTION

In the present paper, some results on common fixed points for a pair of operators in uniform spaces have been presented. After the well-known work of Banach, many authors have generalized his result in different directions. The results due to Edelstein (1961), Kannan (1968), Singh (1969), Srivastava and Gupta (1971), Reich (1971), Rus (1971), Hardy and Rogers (1973), Acharya (1974) and Dass and Khazanchi (1976) are worth mentioning.

Throughout the discussion, (X, U) stands for a sequentially complete separated (Hausdorff) Weil-uniform space defined by a family P of pseudometrics on X. Let

$$V_{(\rho, r)} = \{(x, y) : x, y \in X \text{ and } \rho(x, y) < r, r > 0\}$$

$$G = \{V : V = \bigcap_{i=1}^{n} V_{(\rho_i, r_i)}, \rho_i \in P, r_i > 0, i = 1, \ldots, n\}$$

and

$$\alpha V = \bigcap_{i=1}^{n} V_{(\rho_i, \alpha r_i)}, \rho_i \in P, r_i > 0, \alpha > 0, i = 1, \ldots, n.$$

For the definitions and notations, the reader is referred to Bourkaki (1966) and Kelley (1955).

2. PRELIMINARY RESULTS

We use the following well-known results (cf. Acharya 1974).

Lemma 2.1 — If $V \in G$ and $\alpha, \beta > 0$, then

(i) $\alpha(\beta V) = (\alpha\beta) V$,

(ii) $\alpha V \circ \beta V \subset (\alpha + \beta) V$,

(iii) $\alpha V \subset \beta V$ for $\alpha < \beta$.

Lemma 2.2 — Let ρ be any pseudometric on X and $\alpha, \beta > 0$.

If $(x, y) \in \alpha V_{(\rho, r_1)} \circ \beta V_{(\rho, r_2)}$

then $\rho(x, y) < \alpha r_1 + \beta r_2$.

Lemma 2.3 — If $V \in G$ is arbitrary, there is a pseudometric p on X, such that

$$V = V_{(p, 1)}.$$

This p is called the Minkowski's pseudometric of V.

3. Results

In this section, we obtain some sufficient conditions for the existence of a unique common fixed point of a pair of operators in uniform spaces. It is seen that a result due to the author (Mishra 1977) becomes a corollary here.

3.1 *Theorem* — Let T_1 and T_2 be two operators on X, such that for $V_i \in G$ ($i = 1, \ldots, 5$), $x, y \in X$, $(x, T_1x) \in V_1$, $(y, T_2y) \in V_2$, $(x, T_2y) \in V_3$, $(y, T_1x) \in V_4$ and (x, y) V_5, we have

$$(T_1x, T_2y) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \quad \ldots(3.1.1)$$

where

$$\alpha_i \geqslant 0 \; (i = 1, \ldots, 5), \; \sum_{i=1}^{5} \alpha_i < 1,$$

$$\alpha_1 = \alpha_2 \text{ or } \alpha_3 = \alpha_4. \quad \ldots(3.1.2)$$

Then, T_1 and T_2 have a unique common fixed point.

Proof : Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X as follows

$x_{2n+1} = T_1(x_{2n})$, $x_{2n+2} = T_2(x_{2n+1})$

$n = 0, 1, 2, \ldots$.

Let $V \in G$ be arbitrary. Denote by p a Minkowski's pseudometric of V. Let $x, y \in X$. We take $p(x, T_1x) = s_1$, $p(y, T_2y) = s_2$, $p(x, T_2y) = s_3$, $p(y, T_1x) = s_4$, $p(x, y) = s_5$. Take $\in > 0$. Then, $(x, T_1x) \in (s_1 + \epsilon)V$, $(y, T_2y) \in (s_2 + \epsilon)V$, $(x, T_2y) \in (s_3 + \epsilon)V$, $(y, T_1x) \in (s_4 + \epsilon)V$ and $(x, y) \in (s_5 + \epsilon)V$.

Therefore, by condition (3.1.1), we get

$$(T_1x, T_2y) \in \alpha_1(s_1 + \epsilon) V \circ \alpha_2(s_2 + \epsilon) V \circ \alpha_3(s_3 + \epsilon) V \circ \alpha_4(s_4 + \epsilon) \times V \circ \alpha_5(\rho_5 + \epsilon)V.$$

Using Lemmas 2.1 (i), 2.2 and 2.3, we get

$$p(T_1x, T_2y) < \alpha_1(s_1 + \epsilon) + \alpha_2(s_2 + \epsilon) + \alpha_3(s_3 + \epsilon) \alpha_4(s_4 + \epsilon) + \alpha_5(s_5 + \epsilon)$$

$$< \alpha_1 p(x, T_1x) + \alpha_2 p(y, T_2y) + \alpha_3 p(x, T_2y) + \alpha_4 p(y, T_1x) + \alpha_5 p(x, y) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$p(T_1x, T_2y) \leqslant \alpha_1 p(x, T_1x) + \alpha_2 p(y, T_2y) + \alpha_3 p(x, T_2y) + \alpha_4 p(y, T_1x) + \alpha_5 p(x, y). \qquad \text{...(3.1.3)}$$

We have

$$p(x_1, x_2) = p(T_1x_0, T_2x_1).$$

Using (3.1.3), we get

$$p(x_1, x_2) \leqslant \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3} p(x_0, x_1) = \alpha p(x_0, x_1).$$

where

$$\alpha = \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - \alpha_2 - \alpha_3} < 1.$$

Similarly,

$$p(x_2, x_3) < \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4} p(x_1, x_2) = \alpha\beta p(x_0, x_1)$$

where

$$\beta = \frac{\alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_1 - \alpha_4} < 1.$$

Therefore, in general,

$$p(x_{2n+1}, x_{2n+2}) \leqslant \alpha(\alpha\beta)^n p(x_0, x_1).$$

$$p(x_{2n+2}, x_{2n+3}) \leqslant (\alpha\beta)^{n+1} p(x_0, x_1).$$

Now,

$$\sum_{n=0}^{\infty} p(x_n, x_{n+1}) \leqslant (1 + \alpha) \sum_{n=0}^{\infty} (\alpha\beta)^n p(x_0, x_1).$$

Since $\alpha\beta < 1$, the series on the right converges and so $(x_n, x_{n+1}) \in V$. This shows that $\{x_n\}$ is a cauchy sequence. Since X is sequentially complete, $x_n \to y_0$ for some $y_0 \in X$. To prove that y_0 is a common fixed point of T_1 and T_2, let $V \in G$ be arbitrary and denote by p a Minkowski's pseudometric of V. Then, for any positive integer n, we have

$$p(y_0, T_1y_0) = p(y_0, x_{2n}) + p(T_2x_{2n-1}, T_1y_0).$$

Using (3.1.3), we get

$$(1 - \alpha_1 - \alpha_4) p(y_0, T_1y_0) \leqslant (1 + \alpha_3) p(y_0, x_{2n}) + \alpha_2 p(x_{2n-1}, x_{2n}) + (\alpha_4 + \alpha_5) p(x_{2n-1}, y_0).$$

Letting $n \to \infty$, we obtain

$$p(y_0, T_1 y_0) = 0.$$

This shows that $(y_0, T_1 y_0) \in V$.

Since V is arbitrary and X is Hausdorff, it follows that $T_1 y_0 = y_0$. Similarly, $T_2 y_0 = y_0$.

To prove that y_0 is the unique common fixed point of T_1 and T_2, let Z_0 be such that $T_1 Z_0 = Z_0$. Again, take any $V \in G$.

Then

$$(Z_0, T_1 Z_0) = (Z_0, Z_0) \in V.$$

Also,

$$(y_0, T_2 y_0) = (y_0, y_0) \in V.$$

This implies that

$$(Z_0, y_0) \in (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, V \subset V.$$

This follows from Lemmas 2.2 (ii) and 2.2 (iii).

Since V is arbitrary, it follows that $Z_0 = y_0$.

This completes the proof.

3.2 *Corollary* (Mishra 1977) — Let T be an operator on X, such that for $V_i \in G$ $(i = 1, \ldots, 5)$, $x, y \in X$, $(x, Tx) \in V_1$, $(y, Ty) \in V_2$, $(x, Ty) \in V_3$, $(y, Tx) \in V_4$ and $(x, y) \in V_5$, we have

$$(Tx, Ty) \in \alpha_1 V_1 \circ \alpha_2 V_2 \circ \alpha_3 V_3 \circ \alpha_4 V_4 \circ \alpha_5 V_5 \qquad \ldots(3.2.1)$$

where

$$\alpha_i \geqslant 0, \ \sum_{i=1}^{5} \alpha_i < 1.$$

Then, T has a unique fixed point.

PROOF : We shall just show how this corollary follows as a direct consequence of the theorem proved above. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows :

$$x_n = T(x_{n-1}),\ n = 0, 1, \ldots .$$

Let $V \in G$ be arbitrary. Denote by p a Minkowski's pseudometric of V. Let $x, y \in X$. Then, using the techniques of the above theorem, we get

$$p(Tx, Ty) \leqslant \alpha_1 p(x, Tx) + \alpha_2 p(y, Ty) + \alpha_3 p(x, Ty) + \alpha_4 p(y, Tx) + \alpha_5 p(x, y). \qquad \ldots(3.2.2)$$

By symmetry, we have

$$p(Ty, Tx) \leqslant \alpha_1 p(y, Ty) + \alpha_2 p(x, Tx) + \alpha_3 p(y, Tx) + \alpha_4 p(x, Ty) + \alpha_5 p(x, y). \quad \ldots(3.2.3)$$

Averaging (3.2.2) and (3.2.3), we get

$$p(Tx, Ty) \leqslant \beta_1 p(x, Tx) + \beta_2 p(y, Ty) + \beta_3 p(x, Ty) + \beta_4 p(y, Tz) + \beta_5 p(x, y) \quad \ldots(3.2.4)$$

where

$$\beta_1 = \left(\frac{\alpha_1 + \alpha_2}{2}\right), \ \beta_2 = \left(\frac{\alpha_1 + \alpha_2}{2}\right)$$

$$\beta_3 = \left(\frac{\alpha_3 + \alpha_4}{2}\right), \ \beta_4 = \left(\frac{\alpha_3 + \alpha_4}{2}\right)$$

and

$$\beta_5 = \left(\frac{\alpha_5 + \alpha_5}{2}\right).$$

Therefore, $\beta_1 = \beta_2$, $\beta_3 = \beta_4$ and

$$\sum_{i=1}^{5} \beta_i = \sum_{i=1}^{5} \alpha_i < 1.$$

So, we may assume that $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$. Therefore, by the theorem proved above T has a unique fixed point.

This completes the proof.

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