A REMARK ON DIVISION RINGS WITH INVOLUTION†

by I. N. HERSTEIN, *Department of Mathematics, University of Chicago, Chicago, Illinois* 60637, *USA*

(*Received* 24 *August* 1977)

If D is a division ring with involution and S, K are respectively the symmetric elements and skew elements of D, then it is proved, if $\dim_Z D > 4$, that for any subdivision ring D_0 of D which satisfies $sD_0s^{-1} \subset D_0$ for all $s \neq 0 \in S$, or satisfies $k D_0 k^{-1} \subset D_0$ for all $k \neq 0 \in K$, either $D_0 \subset Z$ or $D_0 = D$ must hold.

The nature of subdivision rings D_0 of a division ring D, which are invariant with respect to certain classes of automorphisms of D, has been studied in a variety of situations. For instance, the Brauer-Cartan-Hua theorem (Jacobson 1964) states that if $xD_0x^{-1} \subset D_0$ for all $x \neq 0 \in D$ then $D_0 = D$ or $D_0 \subset Z$, the centre of D. This was extended by Herstein and Scott (1963) to the following : If N is a normal, non-central subgroup of D and $xD_0x^{-1} \subset D_0$ for all $x \in N$, then $D_0 = D$ or $D_0 \subset Z$. As pointed out by Stuth (1964), one can weaken the hypothesis in this last-cited result to N merely subnormal, non-central in D.

In a different direction, Herstein (1974) showed that if D is a division ring with involution * and $uD_0u^{-1} \subset D_0$ for all unitary elements of D (i.e., $uu^* = 1$), then if char $D \neq 2$, $D_0 \subset Z$ or $D_0 = D$, except for some cases, where $\dim_Z D$ is small.

Let D be a division ring with involution and let $S = \{x \in D \mid x^* = x\}$, $K = \{x \in D \mid x^* = -x\}$ be the set of symmetric and skew elements, respectively, of D. It seems natural to ask if a dichotomy of the Brauer-Cartan-Hua type exists for subdivisions invariant with respect to all the symmetric elements, or all the skew elements, of D. We shall show that the answer is yes provided $\dim_Z R > 4$.

If $\dim_Z D = 4$, the answer need not be yes. For instance, in the quaternions with the usual involution, $S \subset Z$, so, all subdivision rings are invariant re-conjugation by the elements of S. Similarly, if we define the involution * in the quaternions by $(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^* = \alpha_0 + \alpha_1 i - \alpha_2 j - \alpha_3 k$, then $D_0 = \{\alpha_0 + \alpha_1 i\}$ is invariant re-conjugation by all the skew elements, yet $D_0 \neq D$, nor is D_0 central.

Let R be any ring with 1, and let G be the group of invertible elements of R. If $A \subset G$, let $A^n = \{a_1 a_2 \ldots a_n \mid a_i \in A\}$ and let $\hat{A}$ be the subgroup of G generated

†This work was supported by NSF grant MCS-76-06683 at the University of Chicago.

by A. If R has an involution, let S, K be the symmetrics and skews of R, respectively, and let $S_1 = S \cap G$, $K_1 = K \cap G$. (It may happen that K_1 is empty.) The following lemma is useful in our context, and may be useful elsewhere.

Lemma — For any ring with involution, $\hat{S}_1$ is a normal subgroup of G, and if K_1 is not empty $\hat{K}_1^2$ is a normal subgroup of G.

PROOF : The proof is extremely easy, and formal. If $s \in S_1$ and $x \in G$, then $xsx^{-1} = (xsx^*)\, x^{*-1}\, x^{-1})$ and since $xsx^* \in S_1$, $x^{*-1}\, x^{-1} \in S_1$, $xsx^{-1} \in \hat{S}_1$. Thus, $\hat{S}_1$ is normal in G.

Suppose that K_1 is not empty. If $a, b \in K_1$ and $x \in G$ then $xabx^{-1} = (xax^*) \times (x^{*-1}bx^{-1})$, and since xax^* and $x^{*-1}bx^{-1}$ are both in K_1, $xabx^{-1} \in K_1^2$. Thus, $\hat{K}_1^2$ is normal in G.

Note that the same proof actually shows :

(1) For any n, $\hat{S}_1^n$ is normal in G.

(2) If $N = \{xx^* \mid x \in G\}$, then $\hat{N}$ is normal in G.

(3) For any n, $\hat{K}_1^{2n}$ is normal in G (if K_1 is non-empty).

(4) If $L = \{x - x^* \mid x \in R\}$ and $L_1 = L \cap G$ is non-empty, then $\hat{L}_1$ is normal in G.

We now prove the following theorem.

Theorem — Let D be a division ring with involution and let D_0 be a subdivision ring of D. Then,

(1) if $sD_0s^{-1} \subset D_0$ for all $s \neq 0 \in S$, $D_0 = D$ or $D_0 \subset Z$ unless $S \subset Z$ (and so, $\dim_Z D = 4$).

(2) if $aD_0a^{-1} \subset D_0$ for all $a \neq 0 \in K^2$, $D_0 \subset Z$ or $D_0 = D$, unless $\dim_Z D = 4$. (So, if $bD_0b^{-1} \subset D_0$ for all $b \neq 0 \in K$, $D_0 \subset Z$ or $D_0 = D$, unless $\dim_Z D = 4$.)

PROOF : If $K = 0$, then D is commutative and the results are vacuously true. So, we may suppose that $K \neq 0$; hence, K_1 is non-empty.

If $sD_0s^{-1} \subset D_0$ for all $s \neq 0$ in S, then $tD_0t^{-1} \subset D_0$ for all $t \in \hat{S}_1$. By the Lemma, $\hat{S}_1$ is normal in D, so by the result of Herstein and Scott (1963), $D_0 \subset Z$ or $D_0 = D$ unless $\hat{S}_1 \subset Z$. But if $\hat{S}_1 \subset Z$, then $S \subset Z$ and it follows easily that $\dim_Z D = 4$ (*see* Herstein 1976, Theorem 2.16).

If $aD_0a^{-1} \subset D_0$ for all $0 \neq a \in K^2$, then $wD_0w^{-1} \subset D_0$ for all $w \in \hat{K}_1^2$. By the Lemma, $\hat{K}_1^2$ is normal in D, so again by the result of Herstein and Scott, $D_0 \subset Z$ or $D_0 = D$ unless $\hat{K}_1^2 \subset Z$. But then $K^2 \subset Z$ and so, by the proof of Theorem 2.1.10 (Herstein 1976), $\dim_Z D = 4$.

The theorem is now proved.

REFERENCES

Herstein, I. N. (1974). A unitary version of the Brauer-Cartan-Hua theorem. *J. Algebra*, **32**, 555-60.

——— (1976). Rings with Involution. University of Chicago Press, Chicago.

Herstein, I. N., and Scott, W. R. (1963). Subnormal subgroups of division rings, *Can. J. Math.*, **15**, 80-83.

Jacobson, N. (1964). Structure of rings. *Am. math. Soc.*, Colloq. Pub., **37**.

Stuth, C. (1964). A generalization of the Brauer-Cartan-Hua theorem. *Proc. Am. math. Soc.*, **15**, 211-17.