ON NONLINEAR OSCILLATIONS FOR nTH ORDER DELAY INEQUALITIES*

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Recently, Nababan and Noussair (1976) have discussed the oscillation criteria for the nonlinear delay inequality

$$u(t)\,[u''(t) + f(t, u(t), u\,[g(t)])] \leqslant 0, \quad t \geqslant 0.$$

The purpose of this paper is to extend and improve their results to more general nth order inequalities with retarded arguments

$$u(t)Lu(t) \leqslant 0, \quad t \geqslant 0,$$

and

$$u(t)\,Eu(t) \geqslant 0, \quad t \geqslant 0,$$

where

$$Lu(t) = u^{(n)}(t) + f(t, u\,[g_1(t)], \ldots, u\,[g_m(t)]) \text{ for } n \text{ even}$$

and

$$Eu(t) = u^{(n)}(t) - f(t, u\,[g_1(t)], \ldots, u\,[g_m(t)]) \text{ for } n \text{ odd.}$$

1. INTRODUCTION

The present work was inspired by a recent paper of Nababan and Noussair (1976). They gave the oscillation criteria for the non-linear delay inequality

$$u(t)\,[u''(t) + f(t, u(t), u\,[g(t)])] \leqslant 0, \quad t \geqslant 0.$$

The purpose of this paper is to extend and improve their results to more general nth order inequalities with retarded arguments

(A) $$u(t)\,Lu(t) \leqslant 0, \quad t \geqslant 0,$$

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(B) $$u(t)\,Eu(t) \geqslant 0, \quad t \geqslant 0,$$

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Throughout this paper, we assume that conditions (i), (ii), (iii) always hold :

(i) $g_i \in C\,[R_+ \equiv [0, \infty), R_+ - \{0\}]$ with $g_i(t) \leqslant t$ for $t > 0$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \ldots, m$ and $g_*(t) = \min_{1 \leqslant i \leqslant m} g_i(t)$.

(ii) $f \in C\,[R_+ \times R^m, R]$ for $x_i > 0$, $i = 1, 2, \ldots, m$ imply $f(t, x_1, \ldots, x_m)$ is positive and nondecreasing with respect to $x_1, \ldots, x_m$ for all $t \geqslant 0$.

(iii) $f(t, x_1, \ldots, x_m) \leqslant -f(t, -x_1, \ldots, -x_m)$ for $x_i > 0$, $i = 1, 2, \ldots, m$ and all $t \geqslant 0$.

A function is called oscillatory if it has arbitrary large zeros. Otherwise, it is said to be nonoscillatory.

The reader is referred to Erbe (1973), Gollwitzer (1969) and Marušiak (1974).

2. Main Results

To obtain the main results, we need the following lemmas. Lemma 1 is due to Kiguradze (1964, 1965).

Lemma 1 — Let $u(t)$ be a positive n-times continuously differentiable function on an interval $[0, \infty)$. If $u^{(n)}(t)$ is of constant sign and not identically zero for all large t, then there exist a $t_u \geqslant 0$ and an integer k, $0 \leqslant k \leqslant n$, with $n + k$ odd if $u^{(n)}(t) \leqslant 0$, $n + k$ even if $u^{(n)}(t) \geqslant 0$ and such that for every $t \geqslant t_u$

$$k > 0 \text{ implies } u^{(\kappa)}(t) > 0,\ \kappa = 0, 1, \ldots, k - 1$$

$$k \leqslant n - 1 \text{ implies } (-1)^{k+\kappa} u^{(\kappa)}(t) > 0,\ \kappa = k, k + 1, \ldots, n - 1.$$

Lemma 2 — If $u(t)$ is as in Lemma 1 and bounded, $u^{(n)}(t)$ is nonpositive if n even or $u^{(n)}(t)$ is nonnegative if n odd. Then the k, in Lemma 1, is 1.

Proof : Suppose on the contrary that $k > 1$. By Lemma 1 and Taylor's formula, we have

$$u(t) \geqslant u(t_u) + u'(t_u)(t - t_u) + \frac{u''(t_u)}{2!}(t - t_u)^2 + \ldots$$
$$+ \frac{u^{(k-1)}(t_u)}{(k-1)!}(t - t_u)^{k-1}, \text{ for } t > t_u.$$

Thus, $\lim_{t\to\infty} u(t) = \infty$, a contradiction.

Lemma 3 is due to Erbe (1973).

Lemma 3 — Let $g(t) \in C\,[R_+ - \{0\}]$ with $g(t) \leqslant t$ for $t > 0$, $\lim_{t\to\infty} g(t) = \infty$ and $u(t) \in C^2\,[T, \infty)$ satisfy for $t \geqslant T$

$$u(t) > 0, u'(t) > 0 \text{ and } u''(t) \leqslant 0.$$

Then for each $\mu \in (0, 1)$ there exists a $T_\mu \geqslant T$ such that for $t \geqslant T_\mu$

$$u[g(t)] \geqslant \mu u(t) \frac{g(t)}{t}.$$

Theorem 1 — If the differential inequality

$$u^{(n)}(t) + f(t, u[g_*(t)], \ldots, u[g_*(t)]) \leqslant 0 \qquad \ldots(1)$$

has no positive solution $u(t)$ in $[T, \infty)$ for $T > 0$, then inequality (A) is oscillatory.

PROOF : Let $u(t)$ be a nonoscillatory solution of (A). First, we assume that $u(t) > 0$, for $t \geqslant T$, where T is a fixed number. Hence,

$$Lu(t) \leqslant 0 \qquad \ldots(2)$$

for $t \geqslant T$. From condition (i), we can take T large enough, such that $u[g_i(t)] > 0$ for $t \geqslant T$ and $i = 1, 2, \ldots, m$. It follows from (ii), (iii) and (2) that $u^{(n)}(t) \leqslant 0$. By Lemma 1, we have, for T large enough, $u'(t) \geqslant 0$ for $t \geqslant T$, i.e., $u(t)$ is non-decreasing for $t \geqslant T$. Hence, $u[g_i(t)] \geqslant u[g_*(t)]$, $i = 1, 2, \ldots, m$. Using condition (ii), we see easily that $u(t)$ is a positive solution of (1) for $t \geqslant T$ contradicting the hypothesis. Similarly, there cannot exist $T > 0$, such that $u(t)$ is a negative solution of (A) in $[T, \infty)$ or else conditions (ii) and (iii) would imply that $-u(t)$ is a positive solution of (1).

Similarly we have

Theorem 2 — If the differential inequality (1′)

$$u^{(n)}(t) - f(t, u[g_*(t)], \ldots, u[g_*(t)]) \geqslant 0 \qquad \ldots(1')$$

has no positive solution, $u(t)$ in $[T, \infty)$ for $T > 0$, then inequality (B) is oscillatory.

Theorem 3 — If

$$\int^{\infty} [g_*(t)]^{(n-1)\lambda} f(t, c, \ldots, c)\, dt = \pm \infty \qquad \ldots(3)$$

for some $\lambda \in [0, 1]$ and any nonzero constant c, then every bounded solution of (A) is oscillatory.

PROOF : Let $u(t)$ be a nonoscillatory bounded solution of (A). Without loss of generality, we may assume that $u(t) > 0$, for $t \geqslant T$. As in the proof of Theorem 1, we have inequality (2) for $t \geqslant T$. Hence, by Lemma 2,

$$D(t) \equiv \sum_{\kappa=1}^{n-1} (-1)^{1+\kappa} \frac{1}{\kappa!} t^{\kappa} u^{(\kappa)}(t) \geqslant 0.$$

We see easily that

$$u(t) = K(T) + D(t) + (-1)^{n-1} \frac{1}{(n-1)!} \int_T^t s^{n-1} u^{(n)}(s)\, ds$$

$$\geqslant K(T) + \frac{1}{(n-1)!} \int_T^t s^{n-1} f(s, u[g_1(s)], \ldots, u[g_m(s)])\, ds \quad \ldots(4)$$

where $K(T) = u(T) - D(T)$. Since $u'(t) \geqslant 0$ for $t \geqslant T$, $u(t)$ is nondecreasing for $t \geqslant T$. Therefore, (4) and $s \geqslant g_*(s)$ imply

$$u(t) \geqslant K(T) + \frac{1}{(n-1)!} \int_T^t [g_*(s)]^{(n-1)\lambda} f(s, c, \ldots, c)\, ds$$

for some $\lambda \in [0, 1]$, where $c = u(T)$. Thus, by (3), $\lim_{t \to \infty} u(t) = \infty$, a contradiction. This contradiction establishes our theorem.

Similarly, we establish the following theorem.

Theorem 4 — If condition (3) holds, then every bounded solution of (B) is oscillatory.

Using Theorems 3 and 4 and Lemmas 2 and 3, we see easily that Theorem 5 holds.

Theorem 5 — Let

$$\int^{\infty} [g_*(t)]^{(n-1)\lambda} f\left(t, \frac{cg_1(t)}{t}, \ldots, \frac{cg_m(t)}{t} \right) dt = \pm \infty \quad \ldots(5)$$

for any nonzero constant c and some $\lambda \in [0, 1]$. Then every bounded solution of (A) [or (B)] is oscillatory.

Theorem 6 — Let

$$\liminf_{t \to \infty} \frac{g_*(t)}{t} \geqslant \rho > 0 \quad \ldots(6)$$

where ρ is a constant. Then (A) [or (B)] has a bounded nonoscillatory solution if and only if

$$\int^{\infty} [g_*(t)]^{n-1} \, | f(t, c, \ldots, c) | \, dt < \infty \quad \ldots(7)$$

for some $c \neq 0$.

PROOF : We consider only the inequality (A). Conditions (6), (ii) and Theorem 3 show the necessity of (7). If (7) holds, then by (6)

$$\int^{\infty} t^{n-1} \mid f(t, c, \ldots, c) \mid dt < \infty.$$

Hence, the equation

$$u^{(n)}(t) + f(t, u\,[g_1(t)], \ldots, u\,[g_m(t)]) = 0 \qquad \ldots(8)$$

has a bounded nonoscillatory solution by Theorem 1 of Marušiak (1974). But a nonoscillatory solution of (8) is obviously a solution of (A).

From Theorems 3, 4, 5 and 6 we have

Theorem 7 — Every bounded solution of the differential inequality (A) [or (B)] is oscillatory if (3) holds. In addition, if (6) holds, then (3) is also necessary.

REFERENCES

Erbe, L. (1973). Oscillation criteria for second order nonlinear delay equations. *Can. math. Bull.*, **16**, 49-56.

Gollwitzer, H. E. (1969). On nonlinear oscillations for a second order delay equations. *J. math. Analysis Applic.*, **26**, 385-89.

Kiguradze, T. (1964). On the oscillation of solutions of the equation $\frac{d^m u(t)}{dt^m} + \alpha(t) \mid u(t) \mid^n sgn\, u(t) = 0$. *Mat. Sb.*, **65**, 172-87 (Russ.).

Kiguradze, T. (1965). The problem of oscillation of solutions of nonlinear differential equations. *Differential'nye Uravnenija*, **1**, 995-1006 (Russ.).

Marušiak, P. (1974). Oscillation of solutions of the delay differential equation. *Čas. Pest. Mat.*, **99**, 131-41.

Nababan, S., and Noussair, E. S. (1976). Oscillation criteria for second order nonlinear delay inequalities. *Bull. Austral. math. Soc.*, **14**, 331-41.