ON $| V, \lambda |_k$ SUMMABILITY FACTORS OF FOURIER SERIES

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In the present paper, a theorem on $| V, \lambda |_k$ summability factors of Fourier series has been proved which generalizes the well-known results of Pati (1963) and Singh (1967) on the absolute Cesàro summability factors.

§1. Let Σu_n be a given series with the sequence of partial sums $\{s_n\}$ and let $\lambda = \{\lambda_n\}$ be a monotonic non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \leqslant 1$ and $\lambda_1 = 1$. The sequence-to-sequence transformation

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n+1}^{n} s_\nu$$

defines the generalized de la Vallée Poussin means of the sequence $\{s_n\}$ generated by λ.

The series Σu_n is said to be summable $| V, \lambda |$, if the sequence $\{V_n(\lambda)\}$ is of bounded variation, i.e. to say

$$\sum_{n=1}^{\infty} | V_{n+1}(\lambda) - V_n(\lambda) | < \infty \qquad \text{(Leindler 1967).}$$

We say that the series Σu_n is summable $| V, \lambda |_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} | V_{n+1}(\lambda) - V_n(\lambda) |^k < \infty.$$

On taking $\lambda_n = n$, this summability reduces to $| C, 1 |_k$ and for $k = 1$ this is the same as summability $| V, \lambda |$.

§2. Let $f(t)$ be a 2π-periodic and L-integrable function over $(-\pi, \pi)$. We assume, as we may without any loss of generality that

$$\sum_{n=1}^{\infty} A_n(t) \equiv \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

is the Fourier series of $f(t)$.

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\};$$

and

$$S_n(x) = \sum_{\nu=1}^{n} A_\nu(x).$$

§3. In this paper, we prove the following :

Theorem — If $\{\mu_n\}$ is a convex sequence such that

$$\sum \frac{n\mu_n(\log n)^{(1-2\delta)/2}}{\lambda_n^2} < \infty \;(\,|\,\delta\,|\leqslant \tfrac{1}{2}) \qquad \ldots(3.1)$$

$$\sum \frac{n(\log n)^{(1-2\delta)/2}\,\Delta\,\mu_n}{\lambda_n} < \infty \qquad \ldots(3.2)$$

and

$$\int_0^t |\,\phi(u)\,|^k\, du = O\left\{t\left(\log\frac{1}{t}\right)^{\beta}\right\},\; \beta \geqslant 0 \text{ and } 1 \leqslant k \leqslant 2^*, \qquad \ldots(3.3)$$

then the series $\sum \frac{\mu_n A_n(t)}{\{\log(n+1)\}^{(2\beta+2\delta+k-1)/2}}$, at $t = x$ is summable $|\,V, \lambda\,|_k$.

On taking $\lambda_n = n$†, $k = 1$, and $\delta = 0$ in our theorem, we obtain the theorem of Singh (1967) which is an extension of a well-known result of Pati (1963).

§4. We require the following lemmas :

Lemma 1 — If (3.3) holds, then

$$\sum_{\nu=0}^{n} |\,S_\nu(x) - f(x)\,|^k = O\{n(\log n)^{(k/2)+\beta}\},\; \beta \geqslant 0 \text{ and } 1 \leqslant k \leqslant 2. \qquad \ldots(4.1)$$

PROOF : Since the case $k = 1$ of the lemma is due to Cheng (1947), we prove it for $1 < k \leqslant 2$ only.

If (3.3) holds, then by Hölder's inequality, we have

$$\int_0^t |\,\phi(u)\,|\, du \leqslant \left\{\int_0^t |\,\phi(u)\,|^k\, du\right\}^{1/k}.\left\{\int_0^t du\right\}^{1-(1/k)} = O\left\{t\left(\log\frac{1}{t}\right)^{\beta/k}\right\}. \qquad \ldots(4.2)$$

*For $k > 2$, we get another theorem which is not included in the present paper.

†When $\lambda_n = n$, (3.2) immediately follows from (3.1) by virtue of a lemma due to Pati (1954).

Now, taking β/k in place of β in Cheng's lemma (1947), it can be easily proved that if (4.2) holds, then

$$\sum_{\nu=0}^{n} \{S_\nu(x) - f(x)\}^2 = O\{n(\log n)^{1+(2\beta/k)}\}. \qquad \ldots(4.3)$$

Further, applying Hölder's inequality, we have

$$\sum_{\nu=0}^{n} | S_\nu(x) - f(x) |^k \leqslant \{ \sum_{\nu=0}^{n} | S_\nu(x) - f(x) |^2\}^{k/2} . \{ \sum_{\nu=0}^{n} 1\}^{1-(k/2)}, \quad (1 < k < 2)$$

$$= O\{n(\log n)^{(k/2)+\beta}\} \qquad \ldots(4.4)$$

and for $k = 2$, (4.3) and (4.4) are the same.

This completes the proof of the lemma.

Lemma 2 — If (3.3) holds and $T_n(x) = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu A_\nu(x)$,

then

$$\sum_{\nu=1}^{n} | T_\nu(x) |^k = O\{n(\log n)^{(k/2)+\beta}\}.$$

PROOF : Let $\sigma_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} S_\nu(x)$. Then, using lemma 1, we have

$$| \sigma_n(x) - f(x) |^k \leqslant \left\{\frac{1}{n+1} \sum_{\nu=0}^{n} | S_\nu(x) - f(x) | \right\}^k$$

$$\leqslant \frac{1}{(n+1)^k} \left\{ \sum_{\nu=0}^{n} | S_\nu(x) - f(x) |^k\right\} \cdot \left\{ \sum_{\nu=0}^{n} 1\right\}^{k-1}$$

$$= O\{(\log n)^{(k/2)+\beta}\}$$

so that $$\sum_{\nu=1}^{n} | \sigma_\nu(x) - f(x) |^k = O\{n(\log n)^{(k/2)+\beta}\}.$$

Since $T_n(x) = S_n(x) - \sigma_n(x)$, by Minkowski's inequality we have

$$\sum_{\nu=1}^{n} | T_\nu(x) |^k \leqslant [\{ \sum_{\nu=1}^{n} | S_\nu(x) - f(x) |^k\}^{1/k} + \{ \sum_{\nu=1}^{n} | \sigma_\nu(x) - f(x) |^k\}^{1/k}]^k$$

$$= O\{n(\log n)^{(k/2)+\beta}\}.$$

Lemma 3 — If $\{\mu_n\}$ is a convex sequence such that $\sum \frac{n\mu_n}{\lambda_n^2} < \infty$, then

(i) $\sum_{n=1}^{m} \log(n+1) \, . \, \Delta \mu_n = O(1), \; m \to \infty$

(ii) $\sum_{n=1}^{m} n \log(n+1) \, . \, \Delta^2 \mu_n = O(1), \; m \to \infty.$

PROOF : The convergence of $\sum \frac{n\mu_n}{\lambda_n^2}$ implies the convergence of $\sum \frac{\mu_n}{n}$. The remainder of the proof at once follows from the lemmas given by Pati (1954, 1962).

§5. *Proof of the Theorem* — For $k = 1$, the theorem directly follows on taking the series $\sum \frac{\mu_n A_n(t)}{\{\log(n+1)\}^{\beta+\delta}}$ in place of $\sum \mu_n A_n(t)$ and applying the conditions (3.1), (3.2) and (3.3) (with $k = 1$) instead of the set of conditions used by Sharma and Jain (1970) in the proof of their theorem. Therefore, we prove our theorem for $1 < k \leqslant 2$ only.

Let $C_n = V_{n+1}(\lambda; x) - V_n(\lambda; x)$, where $V_n(\lambda; x)$ is the nth de la Vallée Poussin mean of the series

$$\sum \frac{\mu_n A_n(x)}{\{\log(n+1)\}^{(2\beta+2\delta+k-1)/2}}.$$

By an easy computation, we have

$$C_n = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{\nu=n-\lambda_n+2}^{n+1} \{(\lambda_{n+1} - \lambda_n)(\nu - n - 1) + \lambda_n\}$$
$$\times \frac{\mu_\nu A_\nu(x)}{\{\log(\nu+1)\}^{(2\beta+2\delta+k-1)/2}}.$$

Therefore, in order to prove the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} \mid C_n \mid^k < \infty.$$

Let $\overset{(i)}{\underset{n}{\Sigma}}$ be the summation over all n satisfying $\lambda_{n+1} = \lambda_n$ and $\overset{(ii)}{\underset{n}{\Sigma}}$ be the summation over all n where $\lambda_{n+1} > \lambda_n$.

When $\lambda_{n+1} = \lambda_n$, Abel's transformation gives that

$$C_n = \frac{1}{\lambda_{n+1}} \Bigg[\sum_{\nu=n-\lambda_n+2}^{n} \left\{ \sum_{r=1}^{\nu} rA_r(x) \right\} . \, \Delta \left\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \right\} -$$

(*equation continued on p.* 286)

$$- \frac{\mu_{n-\lambda_n+2}}{(n-\lambda_n+2)(\log(n-\lambda_n+3))^{(2\beta+2\delta+k-1)/2}} \left\{ \sum_{r=1}^{n-\lambda_n+1} rA_r(x) \right\}$$

$$+ \frac{\mu_{n+1}}{(n+1)(\log(n+2))^{(2\beta+2\delta+k-1)/2}} \left\{ \sum_{r=1}^{n+1} rA_r(x) \right\} \Bigg]$$

$$= L_n^1 + L_n^2 + L_n^3, \text{ say.}$$

By Minkowski's inequality, it is therefore, sufficient to prove that

$$\sum_n^{(i)} \lambda_n^{k-1} \mid L_n^r \mid^k < \infty \quad \text{for } r = 1, 2, 3.$$

Now, $\displaystyle\sum_n^{(i)} \lambda_n^{k-1} \mid L_n^1 \mid^k$

$$= O(1) \, . \sum_n^{(i)} \frac{1}{\lambda_n} \left[\sum_{\nu=n-\lambda_n+2}^{n} \nu \mid T_\nu(x) \mid . \Delta \left\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \right\} \right]^k$$

$$= O(1) \, . \sum_n^{(i)} \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n+2}^{n} \nu \mid T_\nu(x) \mid^k . \Delta \left\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \right\}$$

$$= O(1) \, . \sum_{\nu=1}^{\infty} \nu \mid T_\nu(x) \mid^k . \Delta \left\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \right\} . \sum_{n=\nu}^{n+\lambda_\nu-1} \frac{1}{\lambda_n}$$

$$= O(1) \, . \sum_{\nu=1}^{\infty} \nu \mid T_\nu(x) \mid^k . \Delta \left\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \right\} .$$

Using Abel's transformation again, by Lemma 2, we easily have

$$\sum_n^{(i)} \lambda_n^{k-1} \mid L_n^1 \mid^k$$

$$= O(1) \, . \sum_{n=1}^{\infty} n^2 (\log n)^{(k/2)+\beta} . \Delta^2 \left\{ \frac{\mu_n}{n(\log(n+1))^{(2\beta+2\delta+k-1)/2}} \right\}$$

$$= O(1) . \sum_{n=1}^{\infty} n(\log n)^{(1-2\delta)/2} . \Delta^2 \mu_n + O(1) . \sum_{n=1}^{\infty} (\log n)^{(1-2\delta)/2} . \Delta \mu_n$$

$$+ O(1) . \sum_{n=1}^{\infty} \frac{\mu_n}{n} (\log n)^{(1-2\delta)/2}$$

$$= O(1) \qquad \ldots(5.1)$$

by Lemma 3(i), 3(ii) and hypothesis (3.1).

Further, applying Abel's transformation and Lemma 2, it is easy to see that

$$\sum_n^{(i)} \lambda_n^{k-1} | L_n^2 |^k + \sum_n^{(i)} \lambda_n^{k-1} | L_n^3 |^k$$

$$= O(1) . \sum_{n=1}^{\infty} | T_n(x) |^k . \frac{\mu_n}{\lambda_n(\log (n+1))^{(2\beta+2\delta+k-1)/2}}$$

$$= O(1) . \sum_{n=1}^{\infty} \frac{n \Delta \mu_n(\log n)^{(1-2\delta)/2}}{\lambda_n} + O(1) . \sum_{n=1}^{\infty} \frac{n\mu_n(\log n)^{(1-2\delta)/2}}{\lambda_n^2}$$

$$+ O(1) . \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} (\log n)^{(-1-2\delta)/2}$$

$$= O(1), \qquad \ldots(5.2)$$

by hypotheses (3.1) and (3.2).

Now, in order to estimate $\sum_n^{(ii)}$ we have, with the aid of Abel's transformation, that

$$| C_n | \leqslant \frac{1}{\lambda_n \lambda_{n+1}} \Big[\sum_{\nu=n-\lambda_n+2}^{n} \nu \, | T_\nu(x) |$$

$$\times \left| \Delta \left\{ (\lambda_n + \nu - n - 1) \frac{\mu_\nu}{\nu(\log (\nu+1))^{(2\beta+2\delta+k-1)/2}} \right\} \right|$$

$$+ (n - \lambda_n + 1) \, | T_{n-\lambda_n+1}(x) | \cdot \frac{\mu_{n-\lambda_n+2}}{(n-\lambda_n+2)(\log(n-\lambda_n+3))^{(2\beta+2\delta+k-1)/2}}$$

$$+ (n+1) \, | T_{n+1}(x) | \cdot \frac{\lambda_n \mu_{n+1}}{(n+1)(\log (n+2))^{(2\beta+2\delta+k-1)/2}}$$

$$= M_n^1 + M_n^2 + M_n^3, \text{ say.}$$

By Minkowski's inequality, it is therefore sufficient to prove that

$$\sum_{n}^{(ii)} \lambda_n^{k-1} \mid M_n^r \mid^k < \infty \quad \text{for } r = 1, 2, 3.$$

Now,

$$\sum_{n}^{(ii)} \lambda_n^{k-1} \mid M_n^1 \mid^k$$

$$\leqslant \sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \Big[\sum_{\nu=n-\lambda_n+2}^{n} \nu \mid T_\nu(x) \mid \Big\{ \lambda_\nu \,.\, \Delta \Big(\frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big)$$

$$+ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big\} \Big]^k$$

$$\leqslant \Big(\Big[\sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \Big\{ \sum_{\nu=n-\lambda_n+2}^{n} \nu \mid T_\nu(x) \mid \lambda_\nu \,.\, \Delta \Big(\frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big) \Big\}^k \Big]^{1/k}$$

$$+ \Big[\sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \Big\{ \sum_{\nu=n-\lambda_n+2}^{n} \lambda_\nu \mid T_\nu(x) \mid \,.\, \frac{\mu_\nu}{\lambda_\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big\}^k \Big]^{1/k} \Big)^k$$

$$= (N_1^{1/k} + N_2^{1/k})^k, \text{ say.}$$

We observe that

$$N_1 = O(1) \,.\, \sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \sum_{\nu=n-\lambda_n+2}^{n} \nu \mid T_\nu(x) \mid^k \lambda_\nu^k$$

$$\times \Delta \Big\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big\}$$

$$= O(1) \,.\, \sum_{\nu=1}^{\infty} \nu \mid T_\nu(x) \mid^k \lambda_\nu^k \,.\, \Delta \Big\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big\}$$

$$\times \sum_{n \geqslant \nu}^{(ii)} \frac{1}{\lambda_n^{k+1}}$$

$$= O(1) \,.\, \sum_{\nu=1}^{\infty} \nu \mid T_\nu(x) \mid^k \,.\, \Delta \Big\{ \frac{\mu_\nu}{\nu(\log(\nu+1))^{(2\beta+2\delta+k-1)/2}} \Big\}$$

$$= O(1), \text{ by (5.1).}$$

And similarly,

$$N_2 = O(1) . \sum_{n}^{(ii)} \frac{1}{\lambda_n^{k+1}} \sum_{\nu=n-\lambda_n+2}^{n} \frac{| T_\nu(x) |^k \lambda_\nu^{k-1} \mu_\nu}{(\log (\nu + 1))^{(2\beta+2\delta+k-1)/2}}$$

$$= O(1) . \sum_{\nu=1}^{\infty} \frac{| T_\nu(x) |^k \mu_\nu}{\lambda_\nu(\log (\nu + 1))^{(2\beta+2\delta+k-1)/2}}$$

$$= O(1), \text{ by (5.2).}$$

Therefore,

$$\sum_{n}^{(ii)} \lambda_n^{k-1} | M_n^1 |^k = O(1).$$

Finally,

$$\sum_{n}^{(ii)} \lambda_n^{k-1} | M_n^2 |^k + \sum_{n}^{(ii)} \lambda_n^{k-1} | M_n^3 |^k$$

$$= O\left[\sum_{n=1}^{\infty} \frac{| T_n(x) |^k \mu_n}{\lambda_n(\log (n + 1))^{(2\beta+2\delta+k-1)/2}} \right]$$

$$= O(1), \text{ by (5.2).}$$

This completes the proof of the theorem.

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