STRATIFIED VISCOUS FLOW OF VARIABLE VISCOSITY BETWEEN A POROUS BED AND MOVING IMPERMEABLE PLATE

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In the present paper, the effects of stratification and slip velocity on the flow of a fluid of variable viscosity over a porous bed have been studied. The flow has been divided in two zones. Flow in zone 1 is governed by the Navier-Stokes equations in the region between the moving impermeable upper plate and the porous bed and that in zone 2 by modified Darcy law. Slip velocity boundary conditions have been used to find the velocity distribution and they are matched at the interface. Fractional decrease in mass flow rate has been calculated and the quantitative effects of slip velocity and stratification factor on the flow have been represented graphically.

1. INTRODUCTION

Beavers and Joseph (1967), Beavers *et al.* (1970) and Rajasekhara *et al.* (1975) studied the flow past a porous medium without stratification. If the variation of viscosity of the fluid is taken into consideration, the results are useful in petroleum industry, as the viscosity in oil varies with temperature. As the flow behaviour of fluids in a petroleum reservoir rock depends to a large extent on the viscous stratification and also on the porous properties of the rock, a technique of core study that can give new or additional information on the characteristics of the rock would provide better understanding of petroleum reservoir performance. Channabasappa and Ranganna (1976) studied the flow of viscous stratified fluids past a permeable bed with the anticipation that stratification may provide a technique for studying the pore size in a porous medium.

The aim of the present study is to investigate the flow of a fluid of variable viscosity over a porous bed under the slip boundary conditions. The flow has been divided into two zones. Zone 1 represents the zone of free flow between the bed and a moving impermeable upper plate. Zone 2 is the zone of Darcy flow below the bed. The basic equations and relevant boundary conditions in the zones have been studied. The problem has been formulated mathematically in Section 2. At the interface, the velocity distributions in these two zones are matched to get a continuous velocity distribution. The fractional decrease in mass flow rate is also obtained and is represented graphically.

2. Mathematical Formulation of the Problem

Here, we have considered the physical model shown in Fig. 1 divided into two zones. Zone 1 is from the impermeable moving plate up to the interface, where the flow is called the free flow and is governed by Navier-Stokes equations. Zone 2 lies below the interface and here the flow is governed by the modified Darcy law.

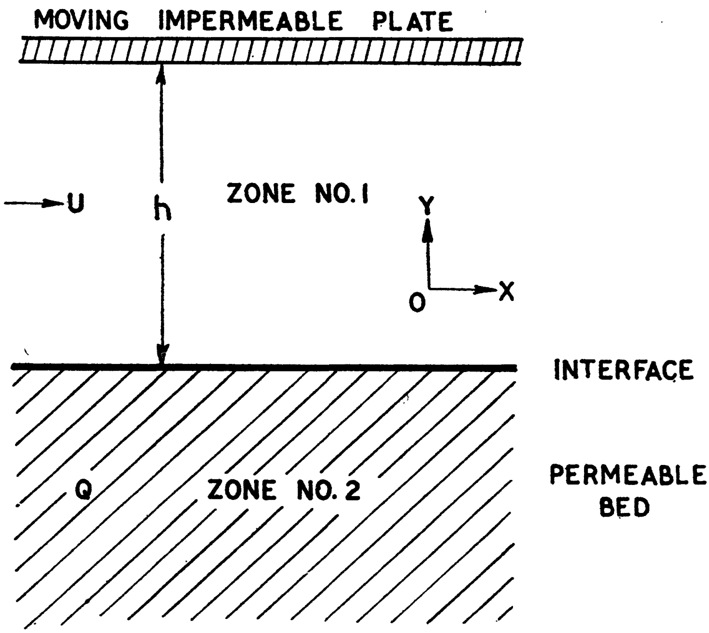

FIG. 1. Physical model.

The basic equations for zone 1 are taken as

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) = \frac{\partial p}{\partial x} \qquad \text{...(1)}$$

where $\mu = \mu_0 e^{-\beta y}$; $\rho = \rho_0 e^{-\beta y}$ ...(2)

and $$\frac{\partial p}{\partial y} = \rho g \qquad \text{...(3)}$$

μ_0 and ρ_0 are the coefficients of viscosity and density respectively at the interface $y = 0$. Also $\beta > 0$ represents the stratification factor.

The basic equations for zone 2 are

$$Q = Q_0 e^{\beta y} \qquad \text{...(4)}$$

where $$Q_0 = -k\left(\frac{\partial p}{\partial x}\right)\mu_0. \qquad \ldots(5)$$

The boundary conditions are

$$U = U_0 \qquad \text{at} \quad y = h \qquad \ldots(6)$$

and $$\frac{\partial U}{\partial y} = \alpha(U_B - Q_0)/\sqrt{k} \qquad \text{at} \quad y = 0 \qquad \ldots(7)$$

where α is the slip parameter; k, the permeability coefficient, which has the dimension of length square; and U_B, the slip velocity at the nominal surface $y = 0$.

The slip boundary condition was first postulated by Beavers and Joseph (1967) and a rigorous theoretical justification for it was given later by Saffman (1971). It is analogous to the slip condition in the kinetic theory of gases.

3. Solution of the Problem

(a) *Velocity distribution* — Here, we shall determine the velocity distribution in zone 1 with the help of boundary conditions (6) and (7). Let us make eqn. (1) dimensionless by using the quantities

$$v = \frac{U}{U_m}; \quad \pi = \frac{p}{\rho_0 U_m^2}; \quad \eta = \frac{y}{h}; \quad \xi = \frac{x}{h}. \qquad \ldots(8)$$

We obtain

$$\frac{\partial^2 v}{\partial \eta^2} - \zeta\frac{\partial v}{\partial \eta} = -Pe^{\zeta\eta}. \qquad \ldots(9)$$

The corresponding boundary conditions are

$$v = \frac{U_0}{U_m} \qquad \text{at } \eta = 1 \qquad \ldots(10)$$

$$\frac{\partial v}{\partial \eta} = a\sigma\left[v_B - \frac{\rho}{\sigma^2}\right] \text{at } \eta = 0 \qquad \ldots(11)$$

where

$$\zeta = \beta h; \quad R = \frac{UD}{\nu_0}; \quad \sigma = \frac{h}{\sqrt{k}}; \quad D = 2h;$$

and $$P = -R\left(\frac{\partial \pi}{\partial \xi}\right)\Big/2. \qquad \ldots(12)$$

Here, ζ is the non-dimensional stratification factor; R, the Reynolds number; and U_m, the maximum velocity of flow.

Solving eqn. (9) with the help of boundary conditions (10) and (11), we obtain

$$\nu = \frac{U_0}{U_m} + \left[\frac{P}{\zeta^2} + \frac{a\sigma}{\zeta}\nu_B - \frac{Pa}{\sigma\zeta}\right](e^{\zeta\eta} - e^{\zeta}) + \frac{P}{\zeta}(e^{\zeta} - \eta e^{\zeta\eta}) \quad ...(13)$$

where ν_B is the dimensionless slip velocity at the nominal surface.

(b) *Fractional decrease in mass flow rate* — To find the quantitative effect of slip velocity and stratification factor, it is required to calculate the mass flow rate in the channel. If M denotes the dimensionless mass flow rate per unit channel width, then

$$M = \int_0^1 e^{-\zeta\eta}\, \nu \, d\eta$$

$$= \frac{1}{\zeta}\left[\frac{U_0}{U_m} - e^{\zeta}\left\{\frac{P}{\zeta^2} + \frac{a\sigma}{\zeta}\nu_B - \frac{P}{\zeta}\left(\frac{a}{\sigma} + 1\right)\right\}\right]\left(1 - \frac{1}{e^{\zeta}}\right)$$

$$+ \frac{P}{\zeta^2} + \frac{a\sigma}{\zeta}\nu_B - \frac{Pa}{\sigma\zeta} - \frac{P}{2\zeta}. \quad ...(14)$$

If the porous bed is replaced by an impermeable rigid plate, then M^*, the dimensionless mass flow rate, is obtained as

$$M^* = \frac{1}{\zeta}\frac{U_0}{U_m}\left(1 - \frac{1}{e^{\zeta}}\right) + \frac{P}{\zeta^2}\left[1 - \frac{\zeta}{2} + \left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{1}{e^{\zeta}}\right)\right] \quad ...(15)$$

From eqns. (14) and (15), we obtain

$$\frac{M}{M^*} = \frac{\frac{1}{\zeta}\left[\frac{U_0}{U_m} - e^{\zeta}\left\{\frac{P}{\zeta^2} + \frac{a\sigma}{\zeta}\nu_B - \frac{P}{\zeta}\left(\frac{a}{\sigma} + 1\right)\right\}\right]\left(1 - \frac{1}{e^{\zeta}}\right) + \frac{P}{\zeta^2} + \frac{a\sigma}{\zeta}\nu_B - \frac{Pa}{\sigma\zeta} - \frac{P}{2\zeta}}{\frac{1}{\zeta}\frac{U_0}{U_m}\left(1 - \frac{1}{e^{\zeta}}\right) + \frac{P}{\zeta^2}\left[1 - \frac{\zeta}{2} + \left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{1}{e^{\zeta}}\right)\right]}. \quad ...(16)$$

The fractional decrease in the mass flow rate through the channel with a permeable lower wall over what it would be if the wall were impermeable is given by

$$\phi = \frac{M}{M^*} - 1 \quad ...(17)$$

or $\phi =$

$$\frac{-\frac{e^{\zeta}}{\zeta}\left[\frac{P}{\zeta^2} + \frac{a\sigma}{\zeta}\nu_B - \frac{P}{\zeta}\left(\frac{a}{\sigma} + 1\right)\right]\left(1 - \frac{1}{e^{\zeta}}\right) + \frac{a\sigma}{\zeta}\nu_B - \frac{Pa}{\sigma\zeta} - \frac{P}{\zeta^2}\left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{1}{e^{\zeta}}\right)}{\frac{1}{\zeta}\frac{U_0}{U_m}\left(1 - \frac{1}{e^{\zeta}}\right) + \frac{P}{\zeta^2}\left[1 - \frac{\zeta}{2} + \left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{1}{e^{\zeta}}\right)\right]}. \quad ...(18)$$

The fractional decrease in mass flow rate, ϕ, as a function of porosity factor, σ, is calculated numerically for different values of a and ζ as under :

(a) $a = 0.01;\quad \zeta = 1$

σ	5	10	15	20	25
ϕ	0.08060	0.16222	0.24384	0.32488	0.40693

(b) $a = 0.01;\quad \zeta = 0.2$

σ	5	10	15	20	25
ϕ	0.66609	0.68393	0.70138	0.71861	0.73381

(c) $a = 0.1;\quad \zeta = 1$

σ	5	10	15	20	25
ϕ	0.86607	1.62432	2.44044	3.25451	4.07092

(d) $a = 0.1;\quad \zeta = 0.2$

σ	5	10	15	20	25
ϕ	0.82096	0.95644	1.16840	1.34213	1.51636

The fractional decrease in mass flow rate, ϕ, as a function of stratification factor, ζ, is also calculated numerically for different values of a and σ as under :

(a_1) $a = 0.01;\quad \sigma = 5$

ζ	0.2	0.4	0.6	0.8	1.0
ϕ	0.66448	0.44079	0.31001	0.18828	0.08060

(b_1) $a = 0.01;\quad \sigma = 10$

ζ	0.2	0.4	0.6	0.8	1.0
ϕ	0.68393	0.49138	0.35721	0.25194	0.16222

(c_1) $a = 0.01;\quad \sigma = 15$

ζ	0.2	0.4	0.6	0.8	1.0
ϕ	0.70134	0.52381	0.40465	0.31542	0.24384

(a_2) $a = 0.1;\quad \sigma = 5$

ζ	0.2	0.4	0.6	0.8	1.0
ϕ	0.68213	0.71187	0.74219	0.77381	0.80607

(b$_2$) $a = 0.1; \quad \sigma = 10$

ζ	0.2	0.4	0.6	0.8	1.0
ϕ	0.89030	1.07105	1.20506	1.38062	1.62432

(c$_2$) $a = 0.1; \quad \sigma = 15$

ζ	0.2	0.4	0.6	0.8	1.0
ϕ	1.16389	1.39024	1.67425	2.02066	2.44044

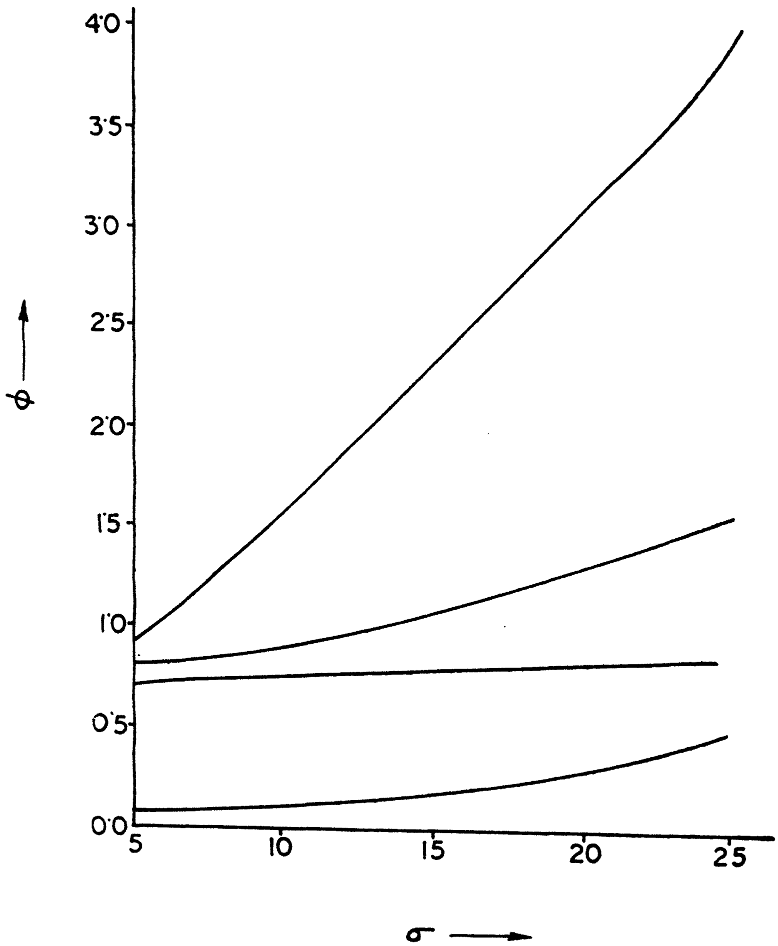

FIG. 2. Fractional decrease in mass flow rate as a function of porosity factor.

4. Discussion

To find the quantitative effect of slip velocity and stratification factor, the fractional decrease in ϕ has been numerically evaluated for different values of σ, a and ζ, as shown in Figs. 2, 3 and 4. From Fig. 2, it is clear that fractional decrease in ϕ increases with increase in the value of σ. Thus, the effect of slip at the bed is to lower the mass flow rate, while the increase in permeability has the reverse effect. It is seen from Fig. 3 that the fractional decrease in mass flow rate relative to the stratification factor ζ decreases numerically with increase in the value of ζ. The fractional decrease in mass flow rate relative to the stratification factor increases numerically with increase in the value of ζ (Fig. 4). The graphs are almost linear in all the three cases, which implies that the viscosity stratification is not favourable to decrease in mass flow rate.

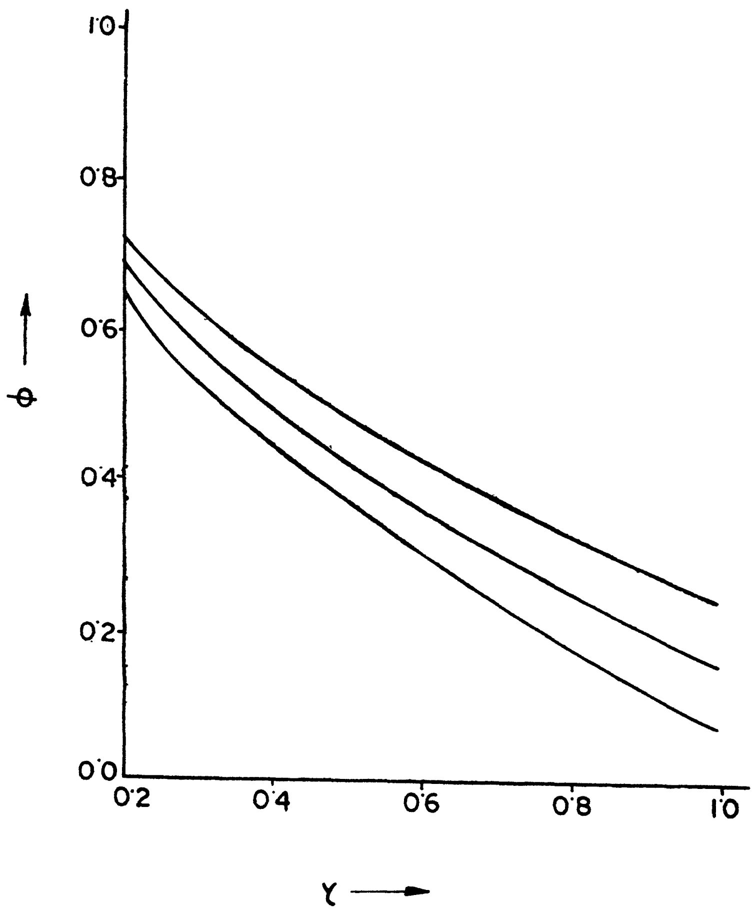

Fig. 3. Fractional decrease in mass flow rate as a function of stratification factor ($a = 0.01$).

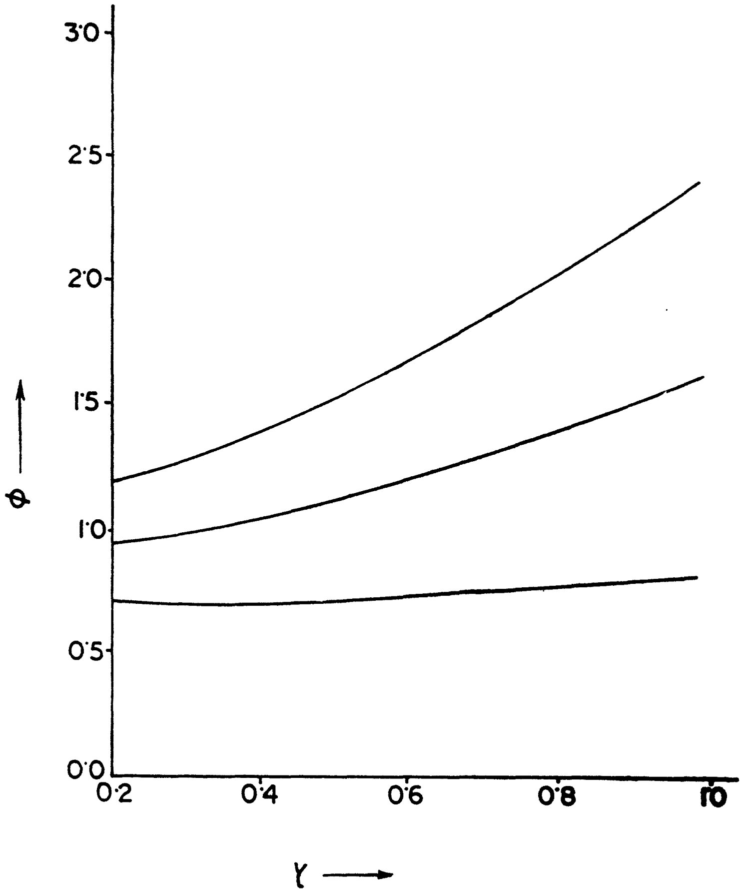

FIG. 4. Fractional decrease in mass flow rate as a function of stratification factor ($a = 0.1$).

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