# ON ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES

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A theorem on absolute Riesz summability of Fourier series with suitable factors has been proved as follows :

Let $\alpha > 0$, $\alpha + \beta < 0$, then

$$\varphi_1(t) \in BV(0, \pi) \Rightarrow \Sigma A_n(x)\, n^{\beta} \in |\, R, e^{n^{\alpha}}, 1 \,|$$

where

$$\varphi_1(t) = \frac{1}{t} \int_0^t \varphi(u)\, du, \quad \varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}$$

and $\Sigma A_n(x)$ is Fourier series defined in usual way. Analogue for conjugate series has also been proved.

## 1. DEFINITIONS AND NOTATIONS

Let $\Sigma\, a_n$ be a given infinite series and $\lambda_n$ is a non-negative strictly increasing monotonic sequence, tending to infinity with $n$. We write for $\omega > \lambda_0$

$$A_\lambda(\omega) \equiv A_\lambda^0(\omega) \equiv \sum_{\lambda_n \leqslant \omega} a_n$$

and for $r > 0$, we write

$$A_{\lambda_n}^r(\omega) \equiv \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^r a_n \equiv \sum_{n \leqslant \omega} \{\lambda(\omega) - \lambda_n\}^r a_n$$

$$= r \int_{\lambda_0}^{\omega} A_\lambda(\tau)\, (\omega - \tau)^{r-1}\, d\tau$$

$$= \int_{\lambda_0}^{\omega} (\omega - \tau)^r\, dA_\lambda(\tau).$$

$A_{\lambda_n}^r(\omega)$ is known as the Riesz sum of type $\lambda_n$ and order $r$ and

$$R_{\lambda_n}^r(\omega) = A_{\lambda_n}^r(\omega)/\omega^r$$

is Rieszian mean of type $\lambda_n$ and order $r$.

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If $R^r_{\lambda_n}(\omega) \to S$ (finite) as $\omega \to \infty$ then we say that $\Sigma\, a_n$ is $(R, \lambda_n, r)$-summable or $\Sigma\, a_n \in (R, \lambda_n, r)$. Further if $R^r_{\lambda_n}(\omega) \in BV(h, \infty)$ for some finite $h > 0$ then $\Sigma\, a_n$ is said to be absolutely summable $(R, \lambda_n, r)$ or symbolically $\Sigma\, a_n \in | R, \lambda_n, r |$.

By definition $| R, \lambda_n, 0 | = | C, 0 |$ i.e. $| R, \lambda_n, 0 |$ is equivalent to absolute convergence of the series $\Sigma\, a_n$.

We use the following notations

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} \qquad \text{...(1.1)}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\} \qquad \text{...(1.2)}$$

$$\varphi_1(t) = \frac{1}{t}\int_0^t \varphi(u)\, du \qquad \text{...(1.3)}$$

$$\psi_1(t) = \frac{1}{t}\int_0^t \psi(u)\, du \qquad \text{...(1.4)}$$

$$e(n) = \exp(n^\alpha) = L(n) \qquad \text{...(1.5)}$$

$$R(\omega, t) = \sum_{e(n) \leqslant \omega} e(n)\, n^\beta e^{int} \qquad \text{...(1.6)}$$

$$Q(\omega, t) = \sum_{n \leqslant \omega} L(n)\, n^\beta e^{int}. \qquad \text{...(1.7)}$$

## 2. Introduction

Let $f$ be $2\pi$-periodic function and $L$-integrable over $(-\pi, \pi)$. The Fourier series of $f$ at a point $t = x$ is

$$\tfrac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_0^\infty A_n(x) \qquad \text{...(2.1)}$$

and conjugate to this series is

$$\sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx) \equiv \sum_1^\infty B_n(x). \qquad \text{...(2.2)}$$

Works of Mohanty (1951) and Chandra (1970, 1972) show that the improvement in the condition affects the type and order of absolute Riesz summability. We further want to lighten the condition and to study the absolute Riesz summability of Fourier series and its conjugate series under De la Vallée Poussin's type of convergence criteria which is certainly weaker than that of Jordan's type of criteria of convergence of Fourier series. Precisely we prove the following:

*Theorem* 1 — For $\alpha > 0$, $\alpha + \beta < 0$

$$\varphi_1(t) \in BV(0, \pi) \Rightarrow \sum_{n=1}^{\infty} A_n(x) \,.\, n^{\beta} \in | R, \exp(n^{\alpha}), 1 | . \qquad \ldots(2.3)$$

*Theorem* 2 — For $\alpha > 0$, $\alpha + \beta < 0$

$$\psi_1(t) \in BV(0, \pi) \Rightarrow \sum_{n=1}^{\infty} B_n(x) \,.\, n^{\beta} \in | R, \exp(n^{\alpha}), 1 | \qquad \ldots(2.4)$$

## 3. Lemmas

*Lemma* 1 — If $\Sigma a_n \in | R, L, r |$, $(r \geqslant 0)$ then $\Sigma a_n \in | R, L, r' |$ for $r' > r$ (Obrechkoff 1928).

*Lemma* 2 — Uniformly in $0 < t \leqslant \pi$, and for large $\omega$, $\alpha > 0$, $\alpha + \beta < 0$ we have,

$$R(\omega, t) \equiv \sum_{\exp(n^{\alpha}) \leqslant \omega} \exp(n^{\alpha}) \,.\, n^{\beta} e^{int} = O\{t^{-1} \,.\, \omega(\log \omega)^{\beta/\alpha}\} \qquad \ldots(3.1)$$

$$Q(\omega, t) \equiv \sum_{n \leqslant \omega} \exp(n^{\alpha}) \,.\, n^{\beta} e^{int} = O\{t^{-1} \exp(\omega^{\alpha}) \,.\, \omega^{\beta}\} \qquad \ldots(3.2)$$

Proof : Let $\exp(m^{\alpha}) \leqslant \omega < \exp\{(m+1)^{\alpha}\}$, we have

$$R(\omega, t) = \sum_{n=1}^{p} + \sum_{n=p+1}^{m} = U + V \text{ say,}$$

where $p$ is an integer such that $n^{\beta} \exp(n^{\alpha})$ is monotonic increasing for $n \geqslant p$, for example

$$p = \left[\left\{-\frac{\beta}{\alpha}\right\}^{1/\alpha} + 1\right]. \text{ Then, we have}$$

$$U = O(1)$$

and

$$V = \sum_{p+1}^{m} n^{\beta} \exp(n^{\alpha})\, e^{int}$$

$$= \{m^{\beta} \exp(m^{\alpha})\} \max_{p \leqslant m' \leqslant m} \left| \sum_{n=m'}^{m} e^{int} \right|$$

$$= O\{t^{-1} m^{\beta} \exp(m^{\alpha})\}.$$

Hence the relation (3.1) follows.

Again

$$Q(\omega, t) = \sum_{n \leqslant \omega} L(n) \,.\, n^{\beta} e^{int}$$

Choosing $m \leqslant \omega < m + 1$, we have

$$Q(\omega, t) = \sum_{n=1}^{p-1} + \sum_{n=p}^{m} = \Sigma_1 + \Sigma_2, \text{ say}$$

where $p$ is again an integer such that $\{L(n)/n^{-\beta}\} \nearrow$ for $n \geqslant p$, for example

$$p = \left[\left(\frac{1}{\alpha}\right)^{1/\alpha}\right] + 1.$$

Then

$$\Sigma_1 = O(1)$$

and

$$\Sigma_2 \leqslant L(m)\, m^{\beta} \max_{p \leqslant m' \leqslant m} \left| \sum_{n=m'}^{m} e^{int} \right|$$

$$= O\{t^{-1} \omega^{\beta} L(\omega)\}.$$

Consequently the relation (3.2) follows.

*Lemma* 3 — Let $\beta < 0$ and

$$A'_n(x) \doteq \int_0^{\pi} \varphi_1(t) \cdot {\cos nt \atop \sin nt}\, dt \qquad \ldots(3.3)$$

then

$$\varphi_1(t) \in BV(0, \pi) \Rightarrow \sum_1^{\infty} A'_n(x)\, n^{\beta} \in |C, 0|. \qquad \ldots(3.4)$$

PROOF : Since $\varphi_1(t) \in BV(0, \pi)$, hence

$$\sum_1^{\infty} |A'_n(x)| \cdot n^{\beta} \cong O(1) \sum_1^{\infty} \frac{1}{n^{1-\beta}} \in |C, 0|.$$

*Proof of Theorem* 1

We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \cos nt\, dt$$

since $\varphi(t) = \varphi_1(t) + t d\varphi_1(t)$, hence

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varphi_1(t) \cos nt\, dt + \frac{2}{\pi} \int_0^{\pi} t d\varphi_1(t) \cos nt$$

$$= P_n + Q_n, \text{ say}.$$

From Lemma 3, $\Sigma\ P_n n^{\beta} \in |\ R, \exp(n^{\alpha}), 0\ |$. Now to prove the theorem, we have to show that

$$I \equiv \int_A^{\infty} \frac{1}{\omega^2} \left| \sum_{\exp(n^{\alpha}) \leqslant \omega} \exp(n^{\alpha})\, n^{\beta} \int_0^t t d\varphi_1(t) \cos nt \right| < \infty.$$

But

$$|\ I\ | \leqslant \int_0^{\pi} t\ |\ d\varphi_1(t)\ | \int_A^{\infty} \frac{1}{\omega^2} \left| \sum_{L(n) \leqslant \omega} \exp(n^{\alpha})\, n^{\beta} \cos nt \right| d\omega$$

since $\int_0^n |\ d\varphi_1(t)\ | < \infty$, hence for our purpose we have only to show that

$$\int_A^{\infty} \omega^{-2}\ |\ R(\omega, t)\ |\ d\omega = O(t^{-1}).$$

By Lemma 2,

$$\int_A^{\infty} \omega^{-2}\ |\ R(\omega, t)\ |\ d\omega = O(t^{-1}) \int_A^{\infty} \omega^{-1}(\log \omega)^{\beta/\alpha}\, d\omega$$

$$= O(t^{-1}) . (\alpha + \beta < 0)$$

This completes the proof of Theorem 1.

*Proof of Theorem* 2

We have

$$B_n(x) = \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin nt\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \psi_1(t) \sin nt\, dt + \frac{2}{\pi} \int_0^{\pi} t d\psi_1(t) \sin nt$$

$$= p_n + q_n, \text{ say.}$$

Since $\Sigma\, p_n n^{\beta} \in |\ C, 0\ |$ and $R(\omega, t) \cong Q(\omega, t)$ hence the proof runs on the lines of proof of Theorem 1. But following proof is also interesting. Now

$$\Sigma\ q_n n^{\beta} \in |\ R, e(n), 1\ |, \text{ iff}$$

$$J = \int_A^{\infty} L'(\omega)\, L^{-2}(\omega) \left| \sum_{n \leqslant \omega} L(n)\, n^{\beta} \int_0^{\pi} t d\psi_1(t) \sin nt \right| < \infty.$$

But

$$| J | \leqslant \int_0^\pi t \, | d\psi_1(t) | \, | \int_A^\infty L'(\omega) \, L^{-2}(\omega) \, | Q(\omega, t) | \, d\omega$$

since $\int_0^\pi | d\psi_1(t) | < \infty$, hence it is sufficient to show that

$$\int_A^\infty L'(\omega) \, . \, L^{-2}(\omega) \, | Q(\omega, t) | \, d\omega = O(t^{-1}).$$

But by Lemma 2,

$$\int_A^\infty L'(\omega) \, L^{-2}(\omega) \, | Q(\omega, t) | \, d\omega = O(t^{-1}) \int_A^\infty L'(\omega) \, L^{-1}(\omega) \, . \, \omega^\beta \, d\omega$$

$$= O(t^{-1}) \int_A^\infty \omega^{\alpha-1+\beta} \, d\omega$$

$$= O(t^{-1}), \ (\alpha + \beta < 0).$$

Hence the proof of Theorem 2 is complete.

## 4. Corollaries

Following are corollaries of Theorems 1 and 2.

*Corollary* 1 — For $\alpha > 0$, $\alpha + \beta < 0$

$$\varphi_1(t) \in BV(0, \pi) \Rightarrow \sum_{n=1}^\infty A_n(x) \, n^\beta \in | C, 1 - \alpha | \, . \qquad \ldots(4.1)$$

*Corollary* 2 — For $\alpha > 0$, $\alpha + \beta < 0$

$$\psi_1(t) \in BV(0, \pi) \Rightarrow \sum_{n=1}^\infty B_n(x) \, n^\beta \in | C, 1 - \alpha | \, . \qquad \ldots(4.2)$$

Proof of Corollaries 1 and 2 : It has been shown (Dikshit 1965) that

$$| R, e^{n^\alpha}, 1 | \Rightarrow | C, 1 - \alpha | , \ \text{for} \ \alpha > 0. \qquad \ldots(4.3)$$

Hence the proofs of Corollaries 1 and 2 follow immediately from theorems 1 and 2 respectively.

*Remark* : It may be remarked that Corollary 1 is a substitute of a well-known result of Bosanquet and Kestelman (1939) who has shown that Fourier series is not $| C, 1 |$ -summable when $\varphi_1(t) \in BV(0, \pi)$ only.

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