THE DISTRIBUTION OF STRESS IN A TRANSVERSELY ISOTROPIC MEDIUM WITH CYLINDRICAL CAVITY AND EXTERNAL CRACK

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This paper deals with the stress field in an infinite transversely isotropic medium having a cylindrical cavity and an external crack surrounding it. Two types of boundary conditions are considered : (i) the cylindrical surface is stress-free, and (ii) the cylindrical lateral surface of the cavity has no radial displacement and is free of shearing stress. The mixed boundary conditions lead to dual integral equations, which are in each case reduced to Fredholm integral equation of the second kind; the latter are solved by the use of Gaussian quadrature formulae. Numerical solutions for some practical materials are carried out and the effect of transverse isotropy on various quantities of interest in fracture mechanics is discussed.

1. INTRODUCTION

Mixed boundary value problems of cylindrical cavity and external cracks seem to have received scant attention so far, even though they appear to be important for the design of various structures, which may not be adequately represented by a two-dimensional model. The earliest three-dimensional analysis of a cylindrical cavity seems to be due to Green and Zernà (1944). Sternberg and Sadowsky (1949) also considered the problem of a cylindrical hole in the plate of arbitrary thickness. The problem of determining the distribution of stress due to an exterior crack in an isotropic infinite elastic medium with a coaxial cylindrical cavity was studied by Srivastava and Narain (1966) using the technique given by Srivastava (1964), by Srivastava and Lee (1972) using the transform technique and by Dhawan (1976) using the transform technique of crack with a cylindrical cavity in a layer. It was shown by Srivastava and Lee (1972) that the kernel of the integral equation and thus the stress intensity factor is independent of the elastic constants. This led us to speculate that anisotropy may not have any effect on the stress intensity factor for the case when the cavity has no radial displacement and is free from shearing stress. In this paper, we treat the above problems for a transversely isotropic material and have shown that transverse isotropy has little effect on the stress intensity factor.

In a transversely isotropic material, the physical property is symmetrical about an axis. It is assumed in this paper, that the axis is parallel to the axis of the cavity. The exact method of analysis for such three-dimensional problems in transversely isotropic materials is usually based on the solution of homogeneous displacement equations of equilibrium in terms of two potential functions, which are harmonic in two distinct spaces, both the spaces being different from the physical space, as shown by Elliot (1948) and Shield (1951). Because two harmonic functions are involved, the mathematical calculations become easier compared to those in the isotropic case where a biharmonic function is involved. Numerical calculations for a particular case are carried out for magnesium and cadmium crystals. In the limit, the isotropic case is derived. The results of limiting isotropic case found to be very close to those of Srivastava and Lee (1972).

2. Basic Equations

Denote as usual the cylindrical coordinates of a point by (r, θ, z), where the z-axis is parallel to the axis of symmetry of the material. We now introduce two potential functions, $\phi_1(r, z_1)$ and $\phi_2(r, z_2)$, where $\phi_1(r, z)$ and $\phi_2(r, z)$ are harmonic in (r, θ, z) space. The axisymmetric torsionless displacements and stresses which satisfy the equilibrium equations are given by the forms with the following two potential functions (Green and Zerna 1944),

$$\left.\begin{aligned} u_r &= \frac{\partial}{\partial r}[\phi_1(r, z_1) + \phi_2(r, z_2)], \\ u_z &= \frac{k_1}{\sqrt{v_1}}\frac{\partial\phi_1(r, z_1)}{\partial z_1} + \frac{k_2}{\sqrt{v_2}}\frac{\partial\phi_2(r, z_2)}{\partial z_2}, \\ u_\theta &= 0 \end{aligned}\right\} \quad \ldots(2.1)$$

and

$$\begin{aligned} \frac{\sigma_{zz}}{c_{44}} &= (1 + k_1)\frac{\partial^2\phi_1(r, z_1)}{\partial z_1^2} + (1 + k_2)\frac{\partial^2\phi_2(r, z_2)}{\partial z_2^2}, \\ \frac{\sigma_{rz}}{c_{44}} &= \frac{(1 + k_1)}{\sqrt{v_1}}\frac{\partial^2\phi_1(r, z_1)}{\partial z_1\partial r} + \frac{(1 + k_2)}{\sqrt{v_2}}\frac{\partial^2\phi_2(r, z_2)}{\partial z_2\partial r}, \\ -\frac{\sigma_{rr}}{c_{44}} &= \frac{1 + k_1}{v_1}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi_1(r, z_1) \\ &\quad + \frac{1 + k_2}{v_2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi_2(r, z_2) \\ &\quad + \frac{(c_{11} + c_{12})}{c_{44}}\cdot\frac{1}{r}\left[\frac{\partial\phi_1(r, z_1)}{\partial r} + \frac{\partial\phi_2(r, z_2)}{\partial r}\right], \\ -\frac{\sigma_{\theta\theta}}{c_{44}} &= \frac{1 + k_1}{v_1}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi_1(r, z_1) \\ &\quad + \frac{1 + k_2}{v_2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi_2(r, z_2) \\ &\quad + \frac{(c_{11} - c_{12})}{c_{44}}\frac{\partial^2}{\partial r^2}[\phi_1(r, z_1) + \phi_2(r, z_2)] \end{aligned} \quad \ldots(2.2)$$

The dimensionless parameters ν_α, k_α ($\alpha = 1, 2$) are dependent upon the elastic constants (c_{ij}), i.e., ν_1 and ν_2 are the roots of the equation

$$c_{44}c_{11}\nu^2 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}]\,\nu + c_{33}c_{44} = 0. \qquad \text{...(2.3)}$$

k_1 and k_2 are defined by

$$k_\alpha = \frac{(c_{11}\nu_\alpha - c_{44})}{(c_{13} + c_{44})}, \; (\alpha = 1, 2), \qquad \text{...(2.4)}$$

and in the equations,

$$z_\alpha = \frac{z}{\sqrt{\nu_\alpha}}, \; (\alpha = 1, 2). \qquad \text{...(2.5)}$$

3. Analysis

We consider the effect of a cylindrical cavity on stress distribution in an elastic transversely isotropic solid with an external crack. The crack is supposed to be on the central plane, which is taken as $z = 0$. The axis of the cavity is assumed to occupy the region $z = 0$, $r \geqslant a > b$. We further assume that the deformation results from the application of the prescribed forces normal to the surface of the crack and that the axial symmetry about the axis of the cavity is preserved. The elastic field is determined by the following boundary conditions :

$$\sigma_{rz}(r, 0) = 0, \qquad b < r < \infty \qquad \text{...(3.1)}$$

$$u_z(r, 0) = 0, \qquad b < r < a \qquad \text{...(3.2)}$$

$$\sigma_{zz}(r, 0) = -\,p_0 p(r), \quad a < r < \infty. \qquad \text{...(3.3)}$$

The two types of boundary conditions on the lateral surface of the cavity which are considered in this paper are as follows :

Case (i) — The cylindrical cavity is inserted in a smooth rigid bore of radius b so that we have on $r = b$

$$\left.\begin{aligned} u_r(b, z) &= 0 \\ \sigma_{rz}(b, z) &= 0. \end{aligned}\right\} \quad z \geqslant 0, \; 0 \leqslant \theta < 2\pi \qquad \text{...(3.4)}$$

Case (ii) — The surface of the cylinder is stress free, so that we have on $r = b$

$$\left.\begin{aligned} \sigma_{rr}(b, z) &= 0 \\ \sigma_{rz}(b, z) &= 0. \end{aligned}\right\} \quad z \geqslant 0, \; 0 \leqslant \theta < 2\pi. \qquad \text{...(3.5)}$$

To solve the mixed boundary value problems, we have two potential functions $\phi_1(r, z_1)$ and $\phi_2(r, z_2)$ to be

$$\phi_1(r, z_1) = \int_0^\infty \alpha^{-1} A_1(\alpha) K_0(\alpha r\sqrt{v_1}) \cos \alpha z \, d\alpha$$

$$+ \int_0^\infty \alpha^{-2} B_1(\alpha) e^{-\alpha z_1} J_0(\alpha r) \, d\alpha \qquad \ldots(3.6)$$

$$\phi_2(r, z_2) = \int_0^\infty \alpha^{-1} A_2(\alpha) K_0(\alpha r\sqrt{v_2} \cos \alpha z \, d\alpha$$

$$+ \int_0^\infty \alpha^{-2} B_2(\alpha) e^{-\alpha z_2} J_0(\alpha r) \, d\alpha. \qquad \ldots(3.7)$$

where $A_1(\alpha)$, $B_1(\alpha)$, $A_2(\alpha)$ and $B_2(\alpha)$ are unknown functions of α to be determined from the boundary conditions. From conditions (3.1), we get

$$B_1(\alpha) = -\frac{1 + k_2}{1 + k_1}\sqrt{\frac{v_1}{v_2}} \, B_2(\alpha). \qquad \ldots(3.8)$$

Using (3.8), we see that the expressions for normal stress and displacement become

$$u_z = -\int_0^\infty \alpha^{-1}\left[\frac{-k_1(1 + k_2)}{(1 + k_1)\sqrt{v_2}} + \frac{k_2}{\sqrt{v_2}}\right] B_2(\alpha) J_0(\alpha r) \, d\alpha$$

$$-\frac{\sigma_{zz}}{c_{44}} = \frac{(1 + k_2)(\sqrt{v_2} - \sqrt{v_1})}{\sqrt{v_2}} \int_0^\infty B_2(\alpha) J_0(\alpha r) \, d\alpha$$

$$-\int_0^\infty \eta[\{(1 + k_1) v_1 A_1(\eta)\} K_0(\eta r\sqrt{v_1})$$

$$+ \{(1 + k_2) v_2 A_2(\eta)\} K_0(\eta r \sqrt{v_2})] \, d\eta. \qquad \ldots(3.9)$$

Hence, from (3.9) and boundary conditions (3.2), (3.3), we see that $B_2(\alpha)$ is a solution of dual integral equations

$$\int_0^\infty B_2(\alpha) J_0(\alpha r) \, d\alpha + \frac{1}{A}\int_0^\infty \eta[(1 + k_1) v_1 A_1(\eta) K_0(\eta r \sqrt{v_1}) + (1 + k_2) v_2$$

$$\times A_2(\eta) K_0(\eta r\sqrt{v_2})] \, d\eta = \frac{p_0 p(r)}{A c_{44}}, \; a < r < \infty \qquad \ldots(3.10)$$

$$\int_0^\infty \alpha^{-1} B_2(\alpha) J_0(\alpha r) \, d\alpha = 0, \qquad b < r < a \qquad \ldots(3.11)$$

with

$$A = \frac{(1+k_2)(\sqrt{v_1} - \sqrt{v_2})}{\sqrt{v_2}}. \qquad \text{...(3.12)}$$

Solution of Integral Equations (3.10) *and* (3.11)

To solve dual integral equations (3.10) and (3.11), let

$$B_2(\alpha) = \alpha \int_0^\infty g(t) \cos(\alpha t)\, dt \qquad \text{...(3.13)}$$

where g is an auxiliary function. With this choice of $B_2(\alpha)$, eqn. (3.11) is automatically satisfied and (3.10) gives

$$g(t) + \frac{1}{A}\int_0^\infty [(1+k_1)\sqrt{v_1}\, e^{-\eta\sqrt{v_1}\, t} \cdot A_1(\eta)$$

$$+ (1+k_2)\sqrt{v_2}\, e^{-\eta\sqrt{v_2}\, t} \cdot A_2(\eta)]\, d\eta$$

$$= \frac{2}{\pi A c_{44}} \int_t^\infty \frac{r p(r)\, dr}{\sqrt{(r^2 - t^2)}}. \qquad \text{...(3.14)}$$

The following integral formulae are used in deriving equation (3.14)

$$\int_t^\infty \frac{r K_0(\eta r \sqrt{v_1})}{\sqrt{(r^2 - t^2)}} = \frac{\pi}{2\eta\sqrt{v_1}}\, e^{-\eta\sqrt{v_1}\, t} \qquad \text{...(3.15)}$$

$$\int_0^\infty \text{Sin}\, \xi t \cdot J_0(\xi r)\, d\xi = \begin{cases} 0, & r > t \\ (t^2 - r^2)^{-1/2}, & r < t. \end{cases} \qquad \text{...(3.16)}$$

In eqn. (3.14), the relation between unknown $g(t)$ and the coefficients $A_1(\eta)$ and $A_2(\eta)$ is determined from the boundary conditions on the surface of the cavity.

Case (i) — From conditions (3.4), we get

$$\int_0^\infty [A_1(\eta)\sqrt{v_1}\, K_1(\eta b \sqrt{v_1}) + A_2(\eta)\sqrt{v_2}\, K_1(\eta b\sqrt{v_2})] \cos(\eta z)\, d\eta$$

$$= -\frac{1+k_2}{\sqrt{v_2}} \int_a^\infty g(t) \int_0^\infty \left[\frac{\sqrt{v_1}}{1+k_1} e^{-\alpha z_1} - \frac{\sqrt{v_2}}{1+k_2} e^{-\alpha z_2}\right]$$

$$\times \cos(\alpha t)\, J_1(\alpha b)\, d\alpha dt \qquad \text{...(3.17)}$$

$$\int_0^\infty [A_1(\eta)\,(1+k_1)\,\sqrt{v_1}\,K_1(\eta b\sqrt{v_1}) + A_2(\eta)\,(1+k_2)\,\sqrt{v_2}\,K_1(\eta b\sqrt{v_2})]\eta$$

$$\times \sin(\eta z)\, d\eta$$

$$= \frac{1+k_2}{\sqrt{v_2}} \int_a^\infty g(t) \int_0^\infty [-e^{-\alpha z_1} + e^{-\alpha z_2}] \cos(\alpha t)\, J_1(\alpha b)\, d\alpha dt \qquad \ldots(3.18)$$

The solutions of which are as follows :

$$A_1(\eta)\sqrt{v_1}\,K_1(\eta b\sqrt{v_1}) + A_2(\eta)\sqrt{v_2}\,K_1(\eta b\sqrt{v_2})$$

$$= \frac{\pi}{2} \cdot \frac{1+k_2}{\sqrt{v_2}} \left[\frac{v_1}{1+k_1} I_1(\eta b\sqrt{v_1}) \right.$$

$$\left. \times M_1 - \frac{v_2}{1+k_2} I_1(\eta b\sqrt{v_2})\, M_2 \right] \qquad \ldots(3.19)$$

$$A_1(\eta)\sqrt{v_1}(1+k_1)\,K_1(\eta b\sqrt{v_1}) + A_2(\eta)\sqrt{v_2}(1+k_2)\,K_1(\eta b\sqrt{v_2})$$

$$= \frac{\pi}{2}\,\frac{1+k_2}{\sqrt{v_2}} [v_1\, I_1(\eta\sqrt{v_1}b)\, M_1 - v_2\, I_1(\eta\sqrt{v_2}b)\, M_2] \qquad \ldots(3.20)$$

where

$$M_i = \int_a^\infty g(t) \exp(-\eta t\sqrt{v_i})\, dt \qquad (i = 1, 2). \qquad \ldots(3.21)$$

Solving (3.19) and (3.20), we get

$$A_1(\eta) = \frac{\pi}{2}\,\frac{1+k_2}{1+k_1} \sqrt{\frac{v_1}{v_2}}\, \frac{I_1(\eta b\sqrt{v_1})}{K_1(\eta b\sqrt{v_1})}\, M_1,$$

$$A_2(\eta) = -\frac{\pi}{2}\, \frac{I_1(\eta b\sqrt{v_2})}{K_1(\eta b\sqrt{v_2})}\, M_2. \qquad \ldots(3.22)$$

Case (ii) — From the conditions (3.5) and applying the same procedure as in case (i), we get the following equations:

$$\left.\begin{aligned} A_1(\eta) &= \frac{\pi}{2}\,\frac{(1+k_2)^2}{v_2} [M_1\triangle_1 + M_2 v_2]/\triangle \\ A_2(\eta) &= \frac{\pi}{2}\,\frac{(1+k_1)(1+k_2)}{\sqrt{v_1 v_2}} [M_1\triangle_1 + M_2\triangle_2]/\triangle \end{aligned}\right\} \qquad \ldots(3.23)$$

where

$$\Delta = (1 + k_1)(1 + k_2)\eta b[\sqrt{v_1}\, K_1(b\eta\sqrt{v_2})\, K_0(\eta b\sqrt{v_2}) - \sqrt{v_2}\, K_1(b\eta\sqrt{v_2}) \times K_0(\eta b\sqrt{v_1})] - \frac{c_{11} - c_{12}}{c_{44}} \sqrt{v_1 v_2}\,(k_1 - k_2)\, K_1(\eta b\sqrt{v_1})\, K_1(\eta b\sqrt{v_2}). \quad \ldots(3.24)$$

$$\Delta_1 = -\eta b\sqrt{v_1 v_2}\,[\sqrt{v_1}\, K_0(\eta b\sqrt{v_2})\, I_1(\eta b\sqrt{v_1}) + \sqrt{v_2}\, K_1(\eta b\sqrt{v_2})\,.\, I_0(\eta b\sqrt{v_1}) + \frac{(c_{11} - c_{12})(k_1 - k_2)}{c_{44}(1 + k_1)(1 + k_2)} v_1 v_2\, K_1(\eta b\sqrt{v_2})\, I_1(\eta b\sqrt{v_1}). \quad \ldots(3.25)$$

$$\Delta_2 = -\eta b\sqrt{v_1 v_2}\,[\sqrt{v_2}\, K_0(\eta b\sqrt{v_1})\, I_1(\eta b\sqrt{v_2}) + \sqrt{v_1}\, K_1(\eta b\sqrt{v_1})\,.\, I_0(\eta b\sqrt{v_2})] + \frac{(c_{11} - c_{12})(k_2 - k_1)}{c_{44}(1 + k_1)(1 + k_2)} v_1 v_2\, K_1(\eta b\sqrt{v_1})\, I_1(\eta b\sqrt{v_2}. \quad \ldots(3.26)$$

Now substituting the values of $A_1(\eta)$ and $A_2(\eta)$ either from (3.22) or (3.23) in eqn. (3.14), we get the Fredholm integral equation of the second kind as follows :

Case (i)

$$g(t) + \frac{\pi(1 + k_2)}{2A\sqrt{v_2}} \int_a^\infty g(u)\, K(u, t)\, du = \frac{2p_0}{\pi A c_{44}} \int_t^\infty \frac{rp(r)\, dr}{\sqrt{(r^2 - t^2)}} \quad \ldots(3.27)$$

where

$$K(u, t) = \int_0^\infty \left[\frac{I_1(\eta b\sqrt{v_1})}{K_1(\eta b\sqrt{v_1})} \cdot v_1\, e^{-\eta\sqrt{v_1}(u+t)} - \frac{I_1(\eta b\sqrt{v_2})}{K_1(\eta b\sqrt{v_2})} \cdot v_2\, e^{-\eta\sqrt{v_2}(u+t)}\right] d\eta. \quad \ldots(3.28)$$

Case (ii)

$$g(t) + \frac{\pi}{2A} \int_0^\infty g(u)\, L(u, t)\, du = \frac{2p_0}{\pi A c_{44}} \int_t^\infty \frac{rp(r)\, dr}{\sqrt{(r^2 - t^2)}} \quad \ldots(3.29)$$

where

$$L(u, t) = \frac{(1 + k_1)(1 + k_2)}{(\sqrt{v_1} - \sqrt{v_2})} \int_0^\infty \frac{1}{\Delta} \Big[\Delta_1 \sqrt{\frac{v_1}{v_2}} \{e^{-\eta\sqrt{v_1}(u+t)} - e^{-\eta\sqrt{v_2}(u+t)}\} + \Delta_2 \sqrt{\frac{v_1}{v_2}} \{e^{-\eta\sqrt{v_1}(u+t)} - e^{-\eta\sqrt{v_2}(u-t)}\} + \sqrt{v_1 v_2} \{e^{-\eta\sqrt{v_1}(u+t)} - e^{-\eta\sqrt{v_1}(u-t)}\} \Big] d\eta. \quad \ldots(3.30)$$

Once the solutions of the integral equations (3·27) and (3.29) are obtained, the stress intensity factor, the deformation of the crack face and the strain energy required to open the crack can be calculated.

4. A Particular Case and Quantities of Physical Interest

Let

$$f(r) = \frac{1}{r^2}\cdot \qquad \ldots(4.1)$$

Then the integral equations (3.27) and (3.29) are reduced to the form

$$g(t) + \frac{\pi(1+k_2)}{2A\sqrt{v_2}} \int_a^\infty g(u)\, K(u, t)\, du = \frac{1}{t} \qquad \ldots(4.2)$$

$$(a < t < \infty)$$

and

$$g(t) + \frac{\pi}{2A} \int_a^\infty g(u)\, L(u, t)\, du = \frac{1}{t} \qquad \ldots(4.3)$$

with kernels $K(u, t)$ and $L(u, t)$ as given in (3.28) and (3.30). Now, we derive quantities of physical interest.

(a) *Normal component of stress* — From (2.2), (3.6) and (3.7), we get

$$\frac{\sigma_{zz}(r, 0)}{c_{44}} = \int_0^\infty [(1+k_1)\, B_1(\alpha) + (1+k_2)\, B_2(\alpha)]\, J_0(\alpha r)\, d\alpha$$

$$- v_1 \int_0^\infty [(1+k_1)\, v_1 \eta A_1(\eta)\, K_0(\eta r\sqrt{v_1}) + (1+k_2)\, v_2 \eta A_2(\eta)$$

$$\times\, K_0(\eta r\sqrt{v_2})]\cdot d\eta. \qquad \ldots(4.4)$$

Now, putting the values of $B_1(\alpha)$ from (3.8) and that of A_1, A_2 from (3.22) for case (i) and from (3.23) for case (ii), we get, after simplification the following :

Case (i)

$$-\frac{\sigma_{zz}(r, 0)}{c_{44}} = \frac{1}{r}\frac{d}{dr} \int_{\max(a, r)}^\infty \frac{t\, g(t)\, dt}{\sqrt{(t^2 - r^2)}} + \frac{1+k_2}{\sqrt{v_2}}\cdot\frac{\pi}{2} \int_a^\infty g(u)\, du \times$$

(equation continued on p. 313)

$$\times \int_0^\infty \Big[v_1^{3/2}\, y\, K_0(yr\sqrt{v_1})\, \frac{I_1(yb\sqrt{v_1})}{K_1(by\sqrt{v_1})}\, e^{-(u+t)y}$$

$$- v_2^{3/2}\, y\, K_0(yr\sqrt{v_2})\,.\, \frac{I_1(yb\sqrt{v_2})}{K_1(yb\sqrt{v_2})}\, e^{-(u+t)y} \Big] dy. \qquad ...(4.5)$$

Case (ii)

$$-\frac{\sigma_{zz}(r,0)}{c_{44}} = \frac{\pi(1+k_1)(1+k_2)}{2\triangle} \int_0^\infty \Big[\sqrt{\frac{v_1}{v_2}}\cdot\eta\cdot(1+k_2)\Big\{\frac{M_1\triangle_1 + M_2 v_2}{\triangle}\Big\}$$

$$\times K_0(\eta r\sqrt{v_1}) + \sqrt{\frac{v_2}{v_1}}\cdot \eta(1+k_2)\Big\{\frac{M_1\triangle_1 + M_2 v_2}{\triangle}\Big\}$$

$$\times K_0(\eta r\sqrt{v_2}) \Big] d\eta \qquad ...(4.6)$$

(b) *Stress intensity factor* — Expression for stress intensity factor is of great importance for workers in fracture mechanics. This expression is :

$$N = \lim_{r\to a} (a-r)^{1/2} \mid \sigma_{zz}(r,0) \mid_{0<r<a} \qquad ...(4.7)$$

which with the help of (4.5) or (4.6), it becomes

$$N = \mu c_{44}\sqrt{\frac{2}{a}}\cdot g(a). \qquad ...(4.8)$$

(c) *Normal component of the displacement* — From (2.1), (3.6), (3.7), (3.8) and (3.13), we get

$$U_z \mid_{z=0} = \frac{k_1 - k_2}{\sqrt{v_2}\,(1+k_1)} \int_a^\infty \frac{g(t)\,dt}{\sqrt{(r^2-t^2)}}. \qquad ...(4.9)$$

Now, (4.9) can be written as

$$U_z \mid_{z=0} = \frac{k_1 - k_2}{\sqrt{v_2}\,(1+k_1)} \Big[\int_a^r \frac{g(r)\,dt}{\sqrt{(r^2-t^2)}} - \int_a^r \frac{g(r)-g(t)}{\sqrt{(r^2-t^2)}}\,dt \Big]$$

$$= \frac{k_1 - k_2}{\sqrt{v_2}\,(1+k_1)} \Big[g(r)\cos^{-1}\Big(\frac{a}{r}\Big) - \int_a^r \frac{g(r)-g(t)}{\sqrt{(r^2-t^2)}}\,dt. \qquad ...(4.10)$$

The above expression for the component of the displacement has the advantage that the singularity at $t = r$ of the integrand has been levelled.

(d) *Crack energy* — The energy required to form the crack is easily computed using the relation

$$W = 2\pi \int_a^\infty \frac{1}{r^2} (U_z \mid_{z=0})\, r\, dr. \qquad \ldots(4.11)$$

Inserting the value for $U_z \mid_{z=0}$, we obtain

$$W = 2\pi \frac{k_1 - k_2}{\sqrt{v_2}\,(1 + k_1)} \int_a^r \frac{1}{r} \int_a^r \frac{g(t)\, dt}{\sqrt{(r^2 - t^2)}}. \qquad \ldots(4.12)$$

Interchanging the order of integration, we get

$$W = \frac{A^2(k_1 - k_2)}{\sqrt{v_2}\,(1 + k_1)} \int_a^\infty \frac{g(t)}{t}\, dt. \qquad \ldots(4.13)$$

5. Numerical Results

Numerical computations have been carried out to find the results of physical interest for magnesium and cadmium crystals and isotropic materials, the last of which is included for comparison and in order to examine the accuracy of numerical calculations. The elastic constants c_{11}, c_{12}, c_{13}, c_{33} and c_{44} of the two crystals have been taken from Huntington (1957) and are presented in Table I.

Theoretically, the analysis cannot be applicable to an isotropic material, because the potential functions ϕ_1 and ϕ_2 are not independent. Then, we have substituded $c_{44} = 0.99997$ for $c_{44} = 1$, which is 0.003% less than the actual isotropic case. We have calculated the limiting isotropic case for $\nu = 0.25$. For the two crystals, the dimensionless values of v_α and k_α ($\alpha = 1, 2$) are computed and are given in Table II.

TABLE I

Elastic constants c_{ij} used (in units of 10, *dynes/cm²).*

	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Isotropy	3.0	1.0	1.0	3.0	0.99997
Magnesium	5.97	2.62	2.17	6.17	1.64
Cadmium	11.0	4.04	3.83	4.69	1.56

TABLE II

Values of v_α *and* k_α ($\alpha = 1, 2$).

	α_1	α_2	k_1	k_2
Isotropy	1.00898	0.991100	1.01351	0.98667
Magnesium	2.05017	0.50411	2.78203	0.35945
Cadmium	1.04862	0.40660	1.85062	0.54036

The kernels given by (3.28) and (3.30) were evaluated at different points using the Laguerre-Gauss method of integration and the integral equations (4.2) and (4.3) by Gauss quadrature formula (Kronrod 1965). The results for stress intensity factor, the deformation of the crack surface and the strain energy required to open the crack are presented in Figs. 1-4. It is interesting to note that when the cylindrical cavity is inserted in a rigid bore,

(a) the stress intensity factor due to the finite radius of the cavity is more pronounced in cadmium than in magnesium, and

(b) the strain energy required to open the crack is found to be in the following order : Magnesium < Cadmium < Isotropic.

For case (b), the numerical results for stress intensity factor are given in Table III. So, for all practical purposes, our limiting isotropic case can be taken to yield the results in the isotropic case.

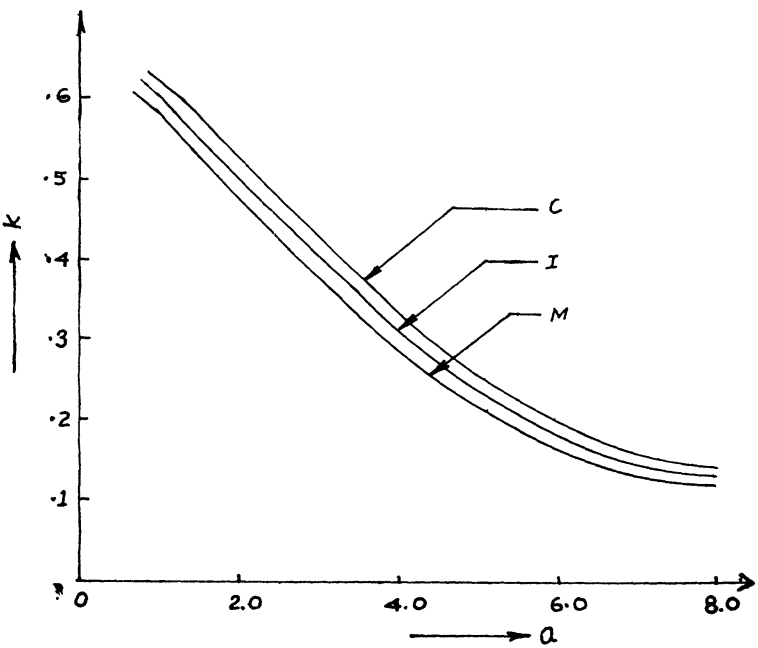

FIG. 1.

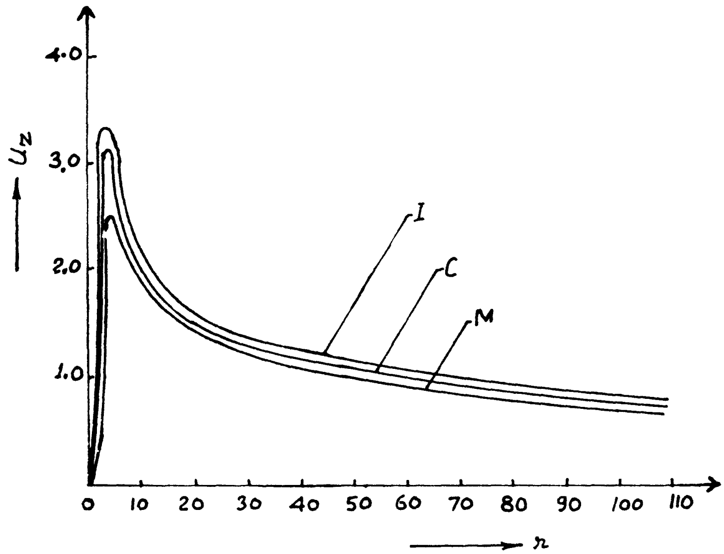

FIG. 2.

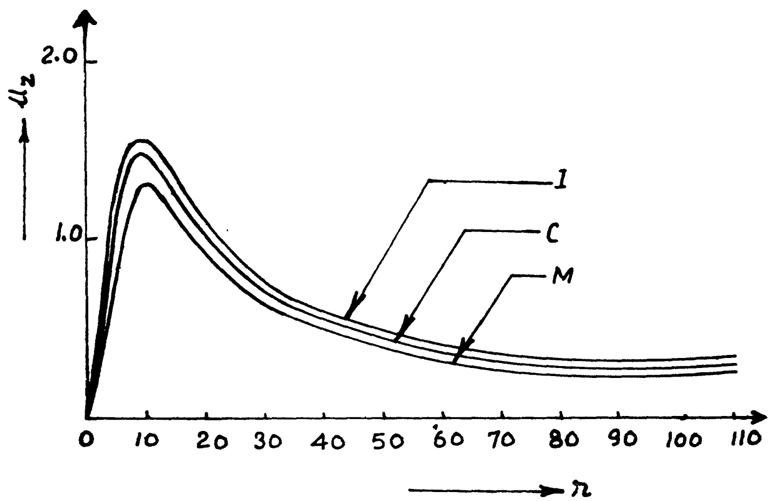

FIG. 3.

TABLE III

Stress intensity factor in case (ii) with a.

$a \rightarrow$	2.0	3.0	4.0	5.0	6.0	7.0	8.0
Cadmium	0.4820	0.2912	0.2381	0.1802	0.1582	0.1035	0.1021
Isotropic	0.4982	0.3013	0.2494	0.2039	0.1664	0.1312	0.1249
Magnesium	0.5103	0.3410	0.2532	0.2218	0.1821	0.1425	0.1309

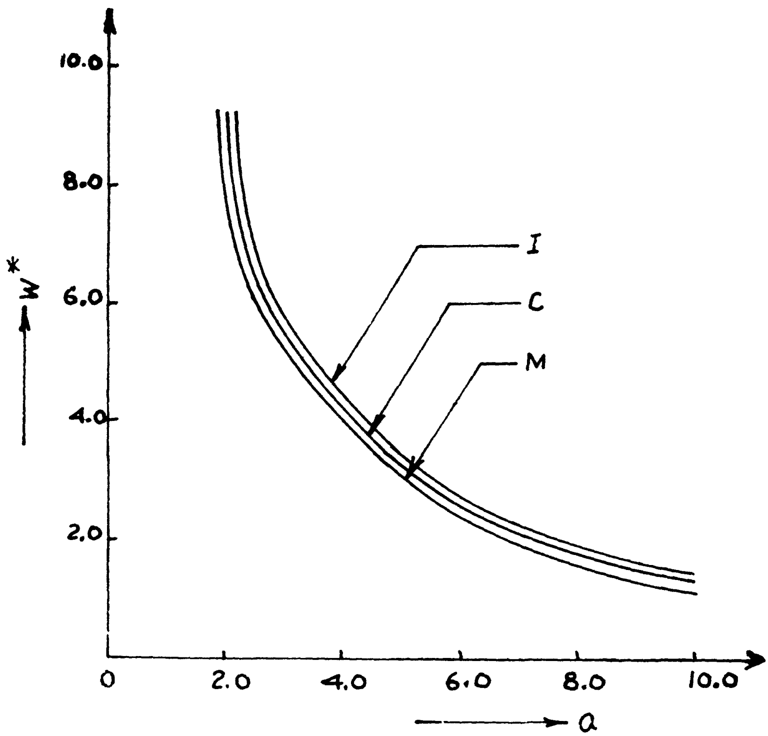

FIG. 4.

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