# AN EXAMPLE IN NON-HAUSDORFF $P$-SPACES

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In this paper an example is given to show that (a) arbitrary intersection of Lindelof sets in a $P$-space need not be expressible as a countable intersection of Lindelof sets (b) Countable intersection of Lindelof sets in a $P$-space need not be expressible as a finite intersection of Lindelof sets (c) finite intersection of Lindelof sets in a $P$-space need not be expressible as the intersection of two Lindelof sets

A topological space is a $P$-space iff every $G_\delta$-set is open. During an analysis of $P$-spaces, the following questions arose : (1) Is each intersection of Lindelof sets in a $P$-space expressible as a countable intersection of Lindelof sets? (2) Is every countable intersection of Lindelof sets in a $P$-space expressible as a finite intersection of Lindelof sets? (3) Is every finite intersection of Lindelof sets in a $P$-space expressible as an intersection of two Lindelof sets?

It is well known that the intersection of Lindelof subsets of a Hausdorff space is again Lindelof if it is a $P$-space. For $P$-spaces which are not Hausdorff, this is clearly not true. Take, for example, an uncountable set $X$. Let $Y = \{a_1, a_2, a_3 \ldots\}$ be a countable subset of distinct elements of $X$. Let us define a topology $\tau$ on $X$ as follows : $A(\subset X)$ is $\tau$-closed if either $A$ is countable or $Y \subset A$. $X$ is clearly a $P$-space (which is $T_1$) such that $Q_i = X - \{a_i\}$ is Lindelof for $i = 1, 2, 3 \ldots$, but $\bigcap_{i=1}^{\infty} Q_i$ is not Lindelof.

We answer all these three questions in the negative by giving the following example.

*Example* — Let $A = [1, \Omega[$. Let us write $X(\alpha; \beta) = \{\beta\} \times \{\alpha\} \times A$, where $\alpha, \beta \in A$. We also take $X(\alpha) = \cup \{X(\alpha; \beta) \mid \beta \leqq \alpha,\ \beta \in A\}$.

Let $X = (\cup \{X(\alpha) \mid \alpha \in A\}) \cup A \cup \{\pi\}$. In this set $X$, we define a topology using fundamental system of open neighbourhoods at each point of $X$.

(a) If $p \in \cup \{X(\alpha) \mid \alpha \in A\}$, then $\{p\}$ is open and this singleton defines a local base at $p$.

(b) Suppose $\alpha \in A$ and $\alpha = \alpha_k + \beta_k$, where $\alpha_k$ and $\beta_k$ are ordinals in $A$. Let us consider the set of all such expressions $\{\alpha_k + \beta_k \mid k \in I(\alpha)\}$ for $\alpha$.

Here, $I(\alpha)$ has been so defined that when $k$ runs over $I(\alpha)$, the set $\{\alpha_k + \beta_k \mid k \in I(\alpha)\}$ gives precisely all possible expressions for $\alpha$ in the form $\alpha_k + \beta_k$ (where, of course, $\alpha_k \geqq \beta_k$). Now, write $Z(\alpha) = X(\alpha) \cup (\cup \{X(\alpha_k; \beta_k) \mid k \in I(\alpha)\})$

$$Y(\alpha) = (\{\alpha\} \cup Z(\alpha)) - P(\alpha)$$

where $P(\alpha)$ is a countable subset of $Z(\alpha)$. Consider the collection of all sets of the form $Y(\alpha)$, as $P(\alpha)$ varies over all possible countable subsets of $Z(\alpha)$. This collection defines a fundamental system of open neighbourhoods at $\alpha$.

(c) The neighbourhood system at $\pi$ is defined as follows : Let $A' \subset A$ be countable, $S(A') = (\cup \{X(\alpha) \mid \alpha \in A'\}) \cup A'$ and $T(A') = X - S(A')$. The collection of all subsets $T(A')$, where $A'$ is a countable subset of $A$ forms the neighbourhood system at $\pi$. Thus, we have defined on $X$ a $P$-space, which is Lindelof and $T_1$, but not Hausdorff.

Consider a set $B \supset X(\alpha; \beta)$. If $B$ is Lindelof, then either $\alpha \in B$ or $\alpha + \beta \in B$. From this it follows that if a set $D \supset X(\alpha)$ and $D$ is Lindelof, then either $\alpha \in D$

or $\{\alpha + 1, \alpha + 2 \ldots, \alpha + \alpha\} \subset D.$

This gives the following three counter examples easily.

(i) Now in $X$, $X - A$ is a subset of $X$, such that $X - A = \cap \{X - \{\alpha\} \mid \alpha \in A\}$, where each $X - \{\alpha\}$ is Lindelof, but it cannot be expressed as a countable intersection of Lindelof subsets of $X$.

(ii) Now write $E = \cup \{X(\alpha) \mid 1 \leqq \alpha < \omega\}$, $F = \cup \{X(\alpha) \cup \{\alpha\} \mid 1 \leqq \alpha < \omega\}$ $E(\beta) = F - \{\beta\}$, where $1 \leqq \beta < \omega$, $E = \cap \{E(\beta) \mid 1 \leqq \beta < \omega\}$, where each $E(\beta)$ is Lindelof. Further, $E$ cannot be expressed as a finite intersection of Lindelof subsets of $X$.

(iii) Let $\alpha$ be finite. We write

$$G(\alpha) = X(\alpha) \cup X(\alpha + 1) \cup X(\alpha + 2) \ldots \cup X(\alpha + \alpha).$$

Let $H(\beta) = G(\alpha) \cup (\{\alpha, \alpha + 1, \alpha + 2, \ldots, 2\alpha + 2\beta\} - \{\alpha + \beta\})$.

Clearly, each $H(\beta)$ is Lindelof. Further, $G(\alpha) = \cap \{H(\beta) \mid 0 \leqq \beta \leqq \alpha\}$. But $G(\alpha)$ cannot be expressed as an intersection of a finite number of Lindelof sets $< \alpha$.

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