A FIXED POINT THEOREM OF A DENSIFYING MAPPING ON A BOUNDED COMPLETE METRIC SPACE

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A fixed point theorem for a densifying mapping satisfying Ciric's (1972) contractive condition has been presented in this note.

INTRODUCTION

Let (X, d) denote a metric space. Recently Ciric proved that if T be a self-mapping of a complete metric space X and if there exist non-negative real valued functions $a_1(x, y)$, $a_2(x, y)$, $a_3(x, y)$ and $a_4(x, y)$ such that

$$\sup \{a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_4(x, y)\} < 1 \qquad \ldots(1)$$

$$d(Tx, Ty) \leqslant a_1(x, y)\, d(x, y) + a_2(x, y)\, d(x, Tx) + a_3(x, y)\, d(y, Ty) + a_4(x, y)\, [d(x, Ty) + d(y, Ty)] \qquad \ldots(2)$$

for all $x, y \in X$, then T has unique fixed point. Ciric (1972) has shown that the condition (2) along with (1) is equivalent to

$$d(Tx, Ty) \leqslant h \max \{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2}\, [d(x, Ty) + d(y, Tx)]\} \qquad \ldots(3)$$

where $0 \leqslant h < 1$.

Now (3) trivially implies

$$d(Tx, Ty) < \max \{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2}\, [d(x, Ty) + d(y, Tx)]\},\ x \neq y. \qquad \ldots(4)$$

The main object of this note is to obtain a fixed point theorem for a continuous densifying mapping satisfying (4). We need the following preliminaries to prove our result.

Definition 1 (Kuratowski 1958, p. 63) — Let A be a bounded subset of a metric space X. By the real number $\alpha(A)$, we denote the infimum of all numbers $\epsilon > 0$ such that A admits a finite covering consisting of subsets with diameter less than ϵ. The number $\alpha(A)$, is usually called the measure of noncompactness of A. It is easily seen that

(i) $0 \leqslant \alpha(A) \leqslant \delta(A)$, where $\delta(A)$ is the diameter of the set A.

(ii) $\alpha(A) = 0$ iff A is precompact.

(iii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$

(iv) If $\bar{A}$ is the closure of A then $\alpha(\bar{A}) = 0 \Leftrightarrow \alpha(A) = 0$ (see Szufla 1968).

Definition 2 (Furi and Vignoli 1969) — Let $T: X \to X$ be a continuous mapping. If for every bounded subset $A \subset X$ with $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$, then T is said to be densifying.

Definition 3 (Ciric 1974) — Let x_0 be a point in X, and T be a mapping X into itself. Then an orbit of x_0 is a sequence

$$\{x_n : x_n = Tx_{n-1},\ n = 1, 2, \ldots\}.$$

We denote the orbit of x_0 by $O(x_0)$.

Definition 4 (Ciric 1974) — A single valued mapping $T: X \to X$ is said to be orbitally continuous if for each $x \in X$, $T^{n_i}x \to \xi$ in X implies $T(T^{n_i}x) \to T\xi$.

Fixed Point Theorems

Theorem 1 — Let T be a mapping of a metric space X into itself such that T satisfies (4). If there exists an orbit, $O(x_0)$ which contains a convergent subsequence on which T is orbitally continuous, then T has unique fixed point.

PROOF: Let $x_0 \in X$ and define $\{x_n\}$ where

$$x_n = T^n x_0,\ x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots .$$

Assume $x_n \neq x_{n+1}$. Then we have a monotone decreasing sequence of non-negative real numbers

$$d(x_0, Tx_0) > d(Tx_0, T^2x_0) > \ldots > d(T^n x_0, T^{n+1}x_0) > \ldots$$

which must converge along with all its subsequences to some number $r \in R$. We have also a convergent subsequence $\{T^{n_i}x_0\}$ in X which converges to some $z_0 \in X$

i.e. $$\lim_{i\to\infty} x_{n_i} = z_0,\quad x_{n_i} = T^{n_i}x_0.$$

We will now show that $z_0 = Tz_0$. Suppose $z_0 \neq Tz_0$. We define

$$z_{n+1} = Tz_n,\quad n = 0, 1, 2, \ldots$$

Then $$d(z_0, z_1) > d(z_1, z_2) > \ldots > d(z_n, z_{n+1}) > \ldots . \qquad \ldots(5)$$

From the orbital continuity of T we get,

$$\begin{aligned} d(z_0, Tz_0) &= d(\lim_{i\to\infty} x_{n_i}, \ T \lim x_{n_i}) \\ &= d(\lim_i x_{n_i}, \ \lim_i x_{n_i+1}) \\ &= \lim_{i\to\infty} d(x_{n_i}, \ x_{n_i+1}) \\ &= r \\ &= \lim_{i\to\infty} d(x_{n_i+n}, \ x_{n_i+n+1}) \\ &= d(z_n, z_{n+1}). \end{aligned} \quad \ldots(6)$$

Hence we get a contradiction from (5) and (6) and it follows from this contradiction that z_0 is a fixed point of T. Unicity of z_0 follows easily from (4). For if z_0, y_0, $z_0 \neq y_0$ be two fixed points of T, then from (4),

$$d(z_0, y_0) < \max \{d(z_0, y_0), 0, 0, d(z_0, y_0)\}$$

which implies $z_0 = y_0$.

Remark 1 : Consider a mapping $T: X \to X$ such that

$$d(Tx, Ty) < ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(x, Tx) + fd(x, y), \quad \text{for each } x \neq y \quad \ldots(7)$$

where $a + b + c + e + f = 1$, $0 \leqslant a, b, c, e, f$. Since (7) implies (4) the following result due to Hardy and Rogers (1973) follows as a Corollary to Theorem 1.

Corollary 1 — Let (X, d) be a compact metric space and T be a continuous mapping of X into itself such that for each pair of distinct elements $x, y \in x$. Let

$$d(Tx, Ty) < ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$$

with $\quad a + b + c + e + f = 1, \ 0 \leqslant a, b, c, e, f$

then T has a unique fixed point.

Corollary 2 — In case X is a compact metric space and $T : X \to X$ such that

$$d(Tx, Ty) < d(x, y), \ x \neq y, \ \ x, y \in X.$$

Then T has a unique fixed point. This is a corollary due to Edelstein (1962).

Theorem 2 — Let (X, d) be a bounded complete metric space and let $T: X \to X$ be a densifying mapping satisfying (4). If for some $x_0 \in X$, the sequence $\{T^n x_0\}$ is bounded then T has a unique fixed point.

PROOF : Let $A = \bigcup_{n=0}^{\infty} \{T^n x_0\}$ and let $\bar{A}$ be the closure of A. Then

$$TA = \bigcup_{n=1}^{\infty} \{T^n x_0\} \subset A,$$

and by the continuity of T, we have $T(\bar{A}) \cup \overline{TA} \subset \bar{A}$. Hence $\bar{A}$ is invariant under T, and is bounded. Next we show that $\bar{A}$ is compact, which by the completeness of X will be true if $\alpha(\bar{A}) = 0$. Suppose $\alpha(\bar{A}) > 0$ or equivalently $\alpha(A) > 0$. Since $A = \{x_0\} \cup T(A)$, we have,

$$\begin{aligned} \alpha(A) &= \max \{\alpha\{x_0\}, \ \alpha(T(A))\} \\ &= \max \{0, \alpha(T(A))\} \\ &= \alpha(T(A)). \end{aligned}$$

But since T is densifying, $\alpha(T(A)) < \alpha(A)$.

This contradiction gives $\alpha(\bar{A}) = 0$ and $\bar{A}$ is compact. Now the space $\bar{A}$ with $T : \bar{A} \to \bar{A}$ satisfies all the assumptions of Theorem 1. Hence by Theorem 1, there is a unique fixed point $z \in \bar{A}$.

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