ON THE DISTURBANCE IN A PORO-ELASTIC MEDIUM BY AN EXPANDING RING AND DISK LOAD

by S. PAUL, *Department of Mathematics, University of Kalyani, Kalyani, Nadia, West Bengal*

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The problem of deformation of a poro-elastic half-space whose surface is disturbed by an expanding ring and disk load, is considered. The displacements derived are expressible in terms of a few single integrals, each of which is identified as a specific wave front. The wave geometry associated with the displacement for the subseismic case has been exhibited. Lastly, the variation with time of normal displacement for a particular case of ring load speed has been shown graphically. The results for the classical case are also included for the sake of comparison.

INTRODUCTION

The dynamical theory of poro-elasticity was established by Biot (1956) and was applied by Deresiewicz (1960) to analyse the effect of boundaries on wave propagation in fluid-filled porous media. Some other problems considered are listed by Paria (1966). The present paper deals with the problem of deformation of a poro-elastic half-space whose surface is disturbed by an expanding ring and disk load. The displacements for all load speeds have been expressed in terms of four displacement potentials as in Deresiewicz and Rice (1962). This has been followed by the application of Laplace and Hankel transform to solve the displacement potentials in the transformed domain. The evaluation of the displacements is effected by Cagniard's technique (Gakenheimer 1971). The wave geometry associated with the displacement for the subseismic case has been exhibited. Lastly, the variation with time of normal displacement for a particular case of ring load speed has been shown graphically. The results for the classical case have also been included for the sake of comparison.

1. PROBLEM AND SOLUTIONS IN THE TRANSFORMED SPACE

We consider the deformation produced by a disk or ring load (normal) of magnitude P_0, which is assumed to originate suddenly from the origin of cylindrical coordinates (r, θ, z) at a constant speed c over the surface of a poro-elastic half-space $z = 0$ with $z > 0$ forming the interior.

The dynamical equations of motion for the fluid-filled porous medium, when the dissipation due to fluid is neglected, are given by Deresiewicz and Rice (1960) as

$$\left.\begin{aligned} P\nabla^2\phi + Q\nabla^2\psi &= \left(\frac{\partial^2}{\partial t^2}\right)(\rho_{11}\phi + \rho_{12}\psi) \\ Q\nabla^2\phi + R\nabla^2\psi &= \left(\frac{\partial^2}{\partial t^2}\right)(\rho_{12}\phi + \rho_{22}\psi) \\ N\left(\nabla^2 - \frac{1}{r^2}\right)H &= \left(\frac{\partial^2}{\partial t^2}\right)(\rho_{11}H + \rho_{12}G) \\ 0 &= \left(\frac{\partial^2}{\partial t^2}\right)(\rho_{12}H + \rho_{22}G) \end{aligned}\right\} \quad \text{and} \quad \ldots(1.1)$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial}{\partial r}\right) + \left(\frac{\partial^2}{\partial z^2}\right)$$

is the Laplacian and the displacement potentials ϕ, ψ, H and G are connected with the solid displacement $\vec{u}(u_r, u_z)$ and the fluid displacement $\vec{U}(U_r, U_z)$ by

$$\left.\begin{aligned} u_r &= \left(\frac{\partial}{\partial r}\right)\phi - \left(\frac{\partial}{\partial z}\right)H, \quad U_r = \left(\frac{\partial}{\partial r}\right)\psi - \left(\frac{\partial}{\partial z}\right)G \\ u_z &= \left(\frac{\partial}{\partial z}\right)\phi + \frac{1}{r}\left(\frac{\partial}{\partial r}\right)(rH), \quad U_z = \left(\frac{\partial}{\partial z}\right)\psi + \frac{1}{r}\left(\frac{\partial}{\partial r}\right)(rG) \end{aligned}\right\} \quad \ldots(1.2)$$

and their relations with the relevant solid stresses σ_{zz}, σ_{rz} and fluid pressure σ are given by

$$\left.\begin{aligned} \sigma_{zz} &= A\nabla^2\phi + Q\nabla^2\psi + 2N\left[\left(\frac{\partial^2}{\partial z^2}\right)\phi + \left(\frac{\partial^2}{\partial z\partial r}\right)H + \frac{1}{r}\left(\frac{\partial}{\partial z}\right)H\right] \\ \sigma_{rz} &= N\left[2\left(\frac{\partial^2}{\partial r\partial z}\right)\phi - \left(\frac{\partial^2}{\partial z^2}\right)H + \left(\frac{\partial^2}{\partial r^2}\right)H + \frac{1}{r}\frac{\partial H}{\partial r} - \frac{H}{r^2}\right] \\ \sigma &= Q\nabla^2\phi + R\nabla^2\psi. \end{aligned}\right\} \quad \ldots(1.3)$$

$P\,(= A + 2N)$, Q, N, R are elastic (non-negative) constants and $\rho_{11}(> 0)$, $\rho_{12}(\leqslant 0)$, $\rho_{22}(\geqslant 0)$ are mass parameters. The boundary conditions at $z = 0$ are taken as

$$\left.\begin{aligned} \sigma_{zz}(r, 0, t) &= -P_0(r, t) \\ \sigma_{rz}(r, 0, t) &= \sigma(r, 0, t) = 0 \end{aligned}\right\} \quad \ldots(1.4)$$

with

$$P_0(r, t) = \begin{cases} \dfrac{F_0}{2\pi r}\,\delta(ct - r) & \text{for ring load} \\ \dfrac{F_0}{\pi(ct)^2}\,H(ct - r) & \text{for disk load} \end{cases} \quad \ldots(1.5)$$

δ and H are the usual Dirac delta function and Heavy-side function respectively. The potentials satisfy the usual initial and regularity conditions. Now applying the Laplace and Hankel transform in the dynamical equation (1.1) and using the boundary conditions (1.5) with the usual initial and regularity conditions, we get the displacements in the Laplace-Hankel transform space. Then inverting the Hankel transform and using the transformation employed by Gakenheimer (1971), we get the displacements in the Laplace transformed space as

$$\bar{u}_j = \sum_{\alpha=1}^{3} \bar{u}_{j\alpha}, \quad \bar{u}_{j\alpha} = \tfrac{1}{2} \int_0^{\infty} \int_{-\infty}^{+\infty} F_{j\alpha} e^{-(s/V_1)(m_j z - iqr)} \, dq \, dw \; (j = r, z) \quad \dots(1.6)$$

where for ring load

$$F_{r1} = k_2 m_0 L q, \qquad F_{z1} = -\, k_2 m_1 m_0 L,$$

$$F_{r2} = -\, k_1 m_0 L q, \qquad F_{z2} = k_1 m_2 m_0 L,$$

$$F_{r3} = 2 m_3 L q, \qquad F_{z3} = 2k^2 F L,$$

and for disk load

$$F_{r1} = 2k_2 M_2 m_0 L' q, \qquad F_{z1} = -\, 2k_2 m_1 m_0 M_2 L',$$

$$F_{r2} = -\, 2k_1 M_2 m_0 L' q, \qquad F_{z2} = 2k_1 m_2 m_0 M_2 L',$$

$$F_{r3} = 4 m_3 F M_2 L' q, \qquad F_{z3} = -\, 4k^2 F M_2 L',$$

with
$$M_2 = m_c - \gamma, \; k^2 = q^2 + w^2, \; L = \left(\frac{F_0}{H^2 m_c R_0 N c}\right), \; L' = \frac{F_0}{H^2 k^2 R_0 N c}$$

$$F = m_2 k_1 - m_1 k_2, \; m_1^2 = k^2 + 1, \; m_2^2 = k^2 + l_1^2, \; m_3^2 = k^2 + l_2^2$$

$$m_c^2 = k^2 + \gamma^2$$

$$l_1 = \frac{V_1}{V_2}, \; l_2 = \frac{V_1}{V_3}, \; \gamma = \frac{V_1}{c}, \; R_0 = m_0^2 - 4k^2 m_3 F, \; k_i = \frac{\xi_i \delta_i^2}{\xi_1 \delta_1^2 - \xi_2 \delta_2^2}$$

$$\xi_i = Q + R\mu_i \; (i = 1, 2), \; \delta_j^2 = \delta_0^2 \Lambda_j, \; \delta_0^2 = \frac{\rho s^2}{H}, \; \Lambda_{1,2} = \frac{B \mp \Delta}{2A},$$

$$\Delta^2 = B^2 - 4Ac, \; \Lambda_3 = \frac{Hc}{N\gamma_{22}}, \; V_j^2 = \left(\frac{H}{\rho}\right)/\Lambda_j, \; A = \sigma_{11}\sigma_{22} - \sigma_{12}^2,$$

$$B = \sigma_{11}\gamma_{22} + \sigma_{22}\gamma_{11} - 2\sigma_{12}\gamma_{12}, \; c = \gamma_{11}\gamma_{22} - \gamma_{12}^2,$$

$$(P, Q, R) = H(\sigma_{11}, \sigma_{12}, \sigma_{22})$$

$$H = P + 2Q + R, \quad (\rho_{11}, \rho_{12}, \rho_{22}) = \rho(\gamma_{11}, \gamma_{12}, \gamma_{22}),$$

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \; \mu_m = \frac{-AA_m + g}{h} \; (m = 1, 2),$$

$$\mu_3 = -\frac{\gamma_{12}}{\gamma_{22}}, \; g = \sigma_{22}\gamma_{11} - \sigma_{12}\gamma_{12}$$

$h = \sigma_{12}\gamma_{22} - \sigma_{22}\gamma_{12}$ $(j = 1, 2, 3)$, V_1, V_2 being the velocities of the dilatational waves of the first and second kind, V_3, the shear wave velocity, and s, the Laplace transform parameter.

2. Exact Inversion of Displacements for $z > 0$

Assuming $V_1 > V_2 > V_3$, the contour integration for each $\bar{u}_{j\alpha}$ is separated into four cases : (i) superseismic[1] ($c > V_1$), (ii) superseismic[2] ($V_1 > c > V_2$), (iii) transeismic ($V_2 > c > V_3$), and (iv) subseismic ($V_3 > c$). The singularities of the integrands in $\bar{u}_{j\alpha}$ in the complex q-plane are the branch points at $q = \pm i(\omega + 1)^{1/2}$, $q = \pm i(\omega^2 + l_1^2)^{1/2}$, $q = \pm i(\omega^2 + l_2^2)^{1/2}$, $q = \pm i(\omega^2 + \gamma^2)^{1/2}$ and the simple poles at

$$q = \pm i(\omega^2 + \gamma_R^2)^{1/2},$$

which are the zeros of Rayleigh function R_0 in (1.7), where $\gamma_R = (V_1/V_R)$, V_R being the Rayleigh wave speed. Now proceeding in the same way as done by Gakenheimer (1971), we get the displacements for all load speed c

$$u_j = \sum_{\alpha=1}^{3} u_{j\alpha}, \; j = r, z \qquad \ldots(2.1)$$

where

$$u_{j1} = H(\tau - 1) \int_0^{T_1} \text{Re}\left[F_{j1}(q_1, \omega)\frac{dq_1}{dt}\right] d\omega + H(c - V_1)\, H(\tau - \tau_{1c})$$

$$H(\tau'_{1c} - \tau)\, H(\phi - \phi_{1c}) \int_{A_{1c}}^{T_{1c}} \text{Re}\left[F_{j1}(q_{1c}, \omega)\frac{dq_{1c}}{dt}\right] d\omega \qquad \ldots(2.2)$$

with

$$A_{1c} = \begin{cases} 0 \text{ for } \tau_{1c} < \tau < 1 \\ T_1 \text{ for } 1 < \tau < \tau'_{1c} \end{cases}$$

$$T_1^2 = \tau^2 - 1, \; T_{1c}^2 = [(\tau - \tau_{1c}) \operatorname{cosec} \phi + \gamma]^2 - \gamma^2, \; \sin \phi_{1c} = \gamma,$$

$$\tau_{1c} = \gamma \sin \phi + (1 - \gamma^2)^{1/2} \cos \phi, \; \tau'_{1c} = (1 - \gamma^2)^{1/2} \sec \phi,$$

$$q_1^{\pm} = i\tau \sin\phi \pm (\tau^2 - \tau_{\omega 1}^2)^{1/2} \cos\phi, \quad \tau = \frac{V_1 t}{\rho}, \; \tau_{\omega 1}^2 = \omega^2 + 1,$$

$$q_{1c} = - i(\tau_{\omega 1}^2 - \tau^2)^{1/2} \cos\phi + i\tau \sin\phi,$$

(ρ, ϕ) are spherical coordinates.

$$\begin{aligned} u_{j_2} &= H(\tau - l_1) \int_0^{T_2} \mathrm{Re}\left[F_{j_2}(q_2, \omega) \frac{dq_2}{dt}\right] d\omega \\ &+ H(c - V_2)\, H(\tau - \tau_{2c})\, H(\tau'_{2c} - \tau)\, H(\phi - \phi_{2c}) \\ &\times \int_{A_{2c}}^{T_{2c}} \mathrm{Re}\left[F_{j_2}(q_{2c}, \omega) \frac{dq_{2c}}{dt}\right] d\omega + H(\tau - \tau_{21})\, H(\tau'_{21} - \tau) \\ &\times H(\phi - \phi_{21}) \int_{A_{21}}^{T_{21}} \mathrm{Re}\left[F_{j_2}(q_{21}, \omega) \frac{dq_{21}}{dt}\right] d\omega \end{aligned} \quad \ldots(2.3)$$

$$A_{2c} = \begin{cases} 0 \ \text{for } \tau_{2c} < \tau < \tau_{21} \\ T_{21} \ \text{for } \tau_{21} < \tau < \tau'_{21} \\ T_2 \ \text{for } \tau'_{21} < \tau < \tau'_{2c} \end{cases} \Bigg\} \; c > V_1, \quad \phi > \phi_{21}$$

$$\quad \begin{cases} 0 \ \text{for } \tau_{2c} < \tau < l_1 \\ T_2 \ \text{for } l_1 < \tau < \tau'_{2c} \end{cases} \quad \begin{cases} V_2 < c < V_1, \; \phi > \phi_{2c} \\ c > V_1, \; \phi_{2c} < \phi < \phi_{21}. \end{cases}$$

$$A_{21} = \begin{cases} 0 \ \text{for } \tau_{21} < \tau < \tau_{2c} \\ T_{2c} \ \text{for } \tau_{2c} < \tau < \tau'_{2c} \\ T_2 \ \text{for } \tau'_{2c} < \tau < \tau'_{21} \end{cases} \Bigg\} \; V_2 < c < V_1, \; \phi > \phi_{2e}$$

$$\quad \begin{cases} 0 \ \text{for } \tau_{21} < \tau < l_1 \\ T_2 \ \text{for } l_1 < \tau < \tau'_{21} \end{cases} \quad \begin{cases} c > V_1, \; \phi > \phi_{21} \\ V_2 < c < V_1, \; \phi_{21} < \phi < \phi_{2c}, \\ \left.\begin{array}{l} V_3 < c < V_2 \\ c < V_3 \end{array}\right\} \phi > \phi_{21} \end{cases}$$

with $T_2^2 = \tau^2 - l_1^2, \; T_{2c}^2 = [(\tau - \tau_{2c}) \operatorname{cosec}\phi + \gamma]^2 - \gamma^2,$

$$T_{21}^2 = [(\tau - \tau_{21}) \operatorname{cosec}\phi + 1]^2 - 1, \; \tau_{2c} = \gamma \sin\phi + (l_1^2 - \gamma^2)^{1/2} \cos\phi,$$

$$\tau'_{2c} = (l_1^2 - \gamma^2) \sec\phi, \; T_{21} = \sin\phi + (l_1^2 - 1)^{1/2} \cos\phi,$$

$$\tau_{21} = (l_1^2 - 1)^{1/2} \sec\phi,$$

$$\sin \phi_{21} = \left(\frac{1}{l_1}\right), \ \sin \phi_{2c} = \left(\frac{\gamma}{l_1}\right),$$

$$q_2^{\pm} = i\tau \sin \phi \pm (\tau^2 - \tau_{\omega 2}^2)^{1/2} \cos \phi, \ \tau_{\omega 2}^2 = \omega^2 + l_1^2,$$

$$q_{2c} = q_{21} = - i(\tau_{\omega 2}^2 - \tau^2)^{1/2} \cos \phi + i\tau \sin \phi.$$

And

$$\begin{aligned} u_{j3} &= H(\tau - l_2) \int_0^{T_3} \mathrm{Re}\left[F_{j3}(q_3, \omega) \frac{dq_3}{dt}\right] d\omega \\ &+ H(c - V_3)\, H(\tau - \tau_{3c})\, H(\tau'_{3c} - \tau) \\ &\times H(\phi - \phi_{3c}) \int_{A_{3c}}^{T_{3c}} \mathrm{Re}\left[F_{j3}(q_{3c}, \omega) \frac{dq_{3c}}{dt}\right] d\omega \\ &+ H(\tau - \tau_{31})\, H(\tau'_{31} - \tau)\, H(\phi - \phi_{31}) \int_{A_{31}}^{T_{31}} \mathrm{Re}\left[F_{j3}(q_{31}, \omega) \frac{dq_{31}}{dt}\right] d\omega \\ &+ H(\tau - \tau_{32})\, H(\tau'_{32} - \tau)\, H(\phi - \phi_{32}) \int_{A_{32}}^{T_{32}} \mathrm{Re}\left[F_{j3}(q_{32}, \omega) \frac{dq_{32}}{dt}\right] d\omega \end{aligned} \quad ...(2.4)$$

with

$$T_3^2 = \tau^2 - l_2^2, \ T_{3c}^2 = [(\tau - \tau_{3c}) \operatorname{cosec} \phi + \gamma]^2 - \gamma^2,$$

$$T_{31}^2 = [(\tau - \tau_{31}) \operatorname{cosec} \phi + 1]^2 - 1, \ T_{32}^2 = [(\tau - \tau_{32}) \operatorname{cosec} \phi + l_1^2] - l_1^2,$$

$$\tau_{31} = \sin \phi + (l_2^2 - 1)^{1/2} \cos \phi, \ \tau'_{3c} = (l_2^2 - \gamma^2)^{1/2} \sec \phi,$$

$$\tau_{32} = l_1 \sin \phi + (l_2^2 - l_1^2)^{1/2} \cos \phi, \ \tau'_{31} = (l_2^2 - 1)^{1/2} \sec \phi,$$

$$\tau_{3c} = \gamma \sin \phi + (l_2^2 - \gamma^2)^{1/2} \cos \phi,$$

$$\sin \phi_{3c} = \left(\frac{\gamma}{l_2}\right), \ \tau \sin \phi_{31} = \left(\frac{1}{l_2}\right), \ \sin \phi_{32} = \frac{l_1}{l_2},$$

$$\tau'_{32} = (l_2^2 - l_1^2)^{1/2} \sec \phi, \ q_3^{\pm} = i\tau \sin \phi \pm (\tau^2 - \tau_{\omega 3}^2)^{1/2} \cos \phi,$$

$$\tau_{\omega 3}^2 = \omega^2 + l_2^2, \ q_{31} = q_{32} = q_{3c} = i\tau \sin \phi - i(\tau_{\omega 3}^2 - \tau^2)^{1/2} \cos \phi.$$

A_{32}, A_{31}, A_{3c} are omitted here, as they are very lengthy. Of course, they are of the same form as A_{21}, A_{20}. The above solutions reduced to the classical case (2), if $V_1 = V_2$. The wave geometry associated with the solid displacement for subseismic case is exhibited in Fig. 1.

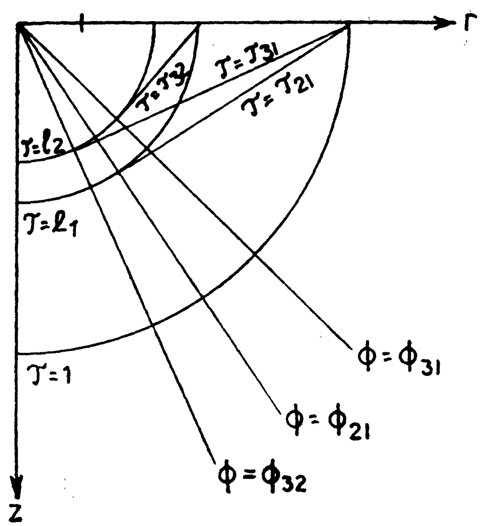

FIG. 1. Wave geometry for subseismic case.

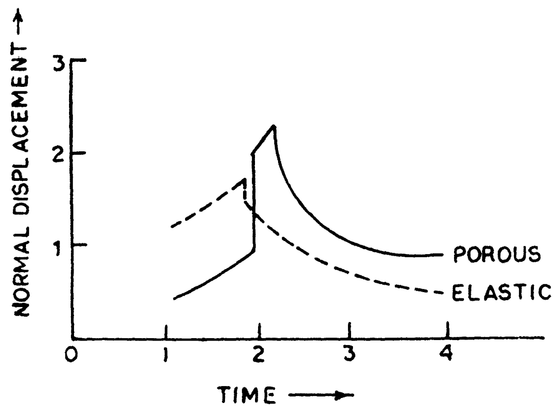

FIG. 2. Comparison between porous and elastic case (ring load case).

The solutions for the case when $V_1 > V_3 > V_2$ can be obtained from these results by interchanging V_2 and V_3. If, in particular, the fluid is absent ($V_2 \to 0$), the results agree with those of Gakenheimer (1971).

3. Numerical Results and Graphical Representations

We have used here the values of parameters as given in Deresiewicz and Rice (1962) which are based on the only experimental result corresponding with sandstone saturated with kerosene. A few results are recorded below for ready reference :

$$k_1 = 0.4588, \quad k_2 = 0.5412, \quad \mu_1 = 0.8904, \quad \mu_2 = -10.2344,$$

$$l_1^2 = 3.8416, \quad l_2^2 = 4.6453, \quad \gamma = 3,$$

$$P = 1.4445, \quad Q = 0.1078, \quad N = 0.3996, \quad R = 0.0473,$$

$$\gamma_{11} = 0.9012, \quad \gamma_{12} = -0.0010, \quad \gamma_{22} = 0.1008.$$

The variation with time with normal displacement for subseismic case for the case $V_1 > V_3 > V_2$ and $\phi = 0°$ is plotted in Fig. 2. The corresponding classical results are also shown in Fig. 2.

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