PROPAGATION OF A MAGNETOGASDYNAMIC SHOCK WAVE AT THE BOUNDARY OF A CONDUCTING GAS

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In this paper we have investigated the nature of phenomena that might occur, when a magnetogasdynamic shock wave propagates through a non-uniform medium of decreasing density and reaches the boundary where density vanishes. The problem has been studied in two parts. The first part deals with a shock wave that just arrives at the boundary of a gas of variable density. If the density distribution near the boundary is approximated by a power law in distance from the boundary, a similarity solution can be locally obtained.

The second part corresponds to the stage in which a mass of gas expands into the vacuum starting with conditions reached at the terminal point of the first stage. The problem is again reduced to that of finding a similarity solution, although of a different kind and no shock is involved.

1. INTRODUCTION

In the present paper we have investigated the nature of phenomena that might occur, when a magnetogasdynamic shock wave propagates through a non-uniform medium of decreasing density and reaches the boundary where the density vanishes.

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The second part corresponds to the stage in which a mass of gas expands into the vacuum starting with the conditions reached at the terminal point of the first stage. The problem is again reduced to that of finding a similarity solution, although of a different kind and no shock is involved.

2. THE FIRST STAGE OF PROPAGATING SHOCK WAVE THROUGH NON-UNIFORM MEDIUM

Let the undisturbed density distribution ρ_0 be equal to zero near the boundary where the phenomena is taken to be one dimensional without any loss of generality.

As shown in Fig. 1, let the x-axis be perpendicular to the boundary $x = 0$ and be directed into the gas. Let the shock wave at $x = X(t)$ propagate in the negative x-direction with the velocity U given by

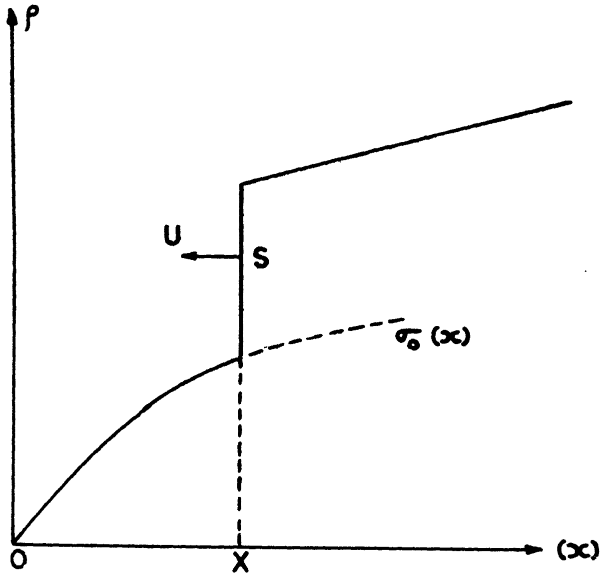

FIG. 1. [$\sigma_0(x)$ be read as $\rho_0(x)$]

$$U = \frac{dX}{dt}. \qquad \ldots(2.1)$$

The time t is taken to be negative until the shock reaches the edge, which means that $X(t) = 0$ when $t = 0$.

Then, the equations governing motion are

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u}{\partial x} \qquad \ldots(2.2)$$

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{h}{\rho}\frac{\partial h}{\partial x} + F_0 \qquad \ldots(2.3)$$

$$\frac{Dh}{Dt} = -h\frac{\partial u}{\partial x} \qquad \ldots(2.4)$$

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0 \qquad \ldots(2.5)$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$, and u, p, ρ, and h are respectively velocity, pressure, density, and magnetic field transverse to the flow behind the shock at time t; F_0 is the body force per unit mass and the gas is assumed to be polytropic with index γ. With the help of eqn. (2.2), eqn. (2.5) can be written as

$$\frac{Dp}{Dt} = -\gamma p \frac{\partial u}{\partial x}. \qquad \ldots(2.6)$$

In front of the shock, in the undisturbed gaseous medium, we have, by our assumption,

$$\rho_0(x) = k_1 x^{\alpha} \qquad \text{...(2.7)}$$

where k_1 and α are positive constants. In the equilibrium state eqn. (2.3) becomes,

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial x} + \frac{h_0}{\rho_0}\frac{\partial h_0}{\partial x} = F_0.$$

If F_0 is zero, a solution of eqns. (2.2) to (2.5) along with (2.7) can be found as a similarity solution of the progressive wave type (Courant and Friedrichs 1948)

$$\left.\begin{aligned} u &= U(x)\, f(\eta) \\ p &= \rho_0(x)\, U^2(x)\, g(\eta) \\ \rho &= \rho_0(x)\, h(\eta) \\ h &= \rho_0^{1/2}(x)\, U(x)\, j(\eta) \end{aligned}\right\} \qquad \text{...(2.8)}$$

where

$$\eta = \left(\frac{X}{x}\right)^{\lambda+1} \text{ varies as } x^{-\lambda-1}t, \quad 0 \leqslant \eta \leqslant 1 \qquad \text{...(2.9)}$$

and the shock velocity U is given by

$$U(X) = k_2 X^{-\lambda} \qquad \text{...(2.10)}$$

k_2 and λ being constants. Accordingly, $U(x)$ in (2.8) is $k_2 x^{-\lambda}$ from (2.10).

This special type of similarity solution (2.8) is selected to fit the singularity at $X = 0$.

Putting the values of (2.8) to (2.10) in the eqns. (2.2) to (2.6), we have the following equations for f, g, h and j.

$$(1-\eta f)\frac{h'}{h} - \eta f' = -\frac{\alpha-\lambda}{(1+\lambda)} f$$

$$(1-\eta f)\frac{g'}{g} - \gamma\eta f' = -\frac{\alpha-(\gamma+2)\lambda}{(1+\lambda)} f$$

$$(1-\eta f) f' - \eta\frac{g'}{h} - \frac{jj'\eta}{h} = \frac{\lambda}{1+\lambda} f^2 - \frac{\alpha-2\lambda}{(1+\lambda)}\frac{g}{h} - \frac{(\alpha-2\lambda)}{2(1+\lambda)}\frac{j^2}{h}$$

$$(1-\eta f)\frac{j'}{j} - \eta f' = \frac{(4\lambda-\alpha)}{2(1+\lambda)} f. \qquad \text{...(2.11)}$$

Since $\lambda > 0$, U becomes large for small x. The velocity of sound c_0 near $x = 0$ on the other hand is small because of small density with the result that the shock becomes very strong there. The Rankine-Hugoniot conditions for strong shock wave at $x = X$ (or $\eta = 1$) are, as in Whitham (1958)

$$\left.\begin{aligned} u &= \frac{2}{\gamma+1} U(X) \\ p &= \frac{2}{\gamma+1} \rho_0(X)\, U^2(X) \\ \rho &= \frac{\gamma+1}{\gamma-1} \rho_0(X) \\ h &= \frac{\gamma+1}{\gamma-1} h_0(X). \end{aligned}\right\} \quad \ldots(2.12)$$

On comparison with (2.8) the above relations give the boundary conditions for f, g, h and j at $\eta = 1$ as follows:

$$f(1) = \frac{2}{\gamma+1},\; g(1) = \frac{2}{\gamma+1},\; h(1) = \frac{\gamma+1}{\gamma-1}$$

$$j(1) = \frac{1}{M_A}\frac{\gamma+1}{\gamma-1} \quad \ldots(2.13)$$

where

$$M_A = \frac{U}{c}.$$

Now we have a system of ordinary differential equations (2.11) and boundary conditions (2.13). Taking $M_A = 10$ as taken by Pai (1958), a solution of eqns. (2.11) in the region $0 \leqslant \eta \leqslant 1$ starting from the initial values (2.13) at $\eta = 1$ has been obtained with the help of the following approximate formula

$$\psi(1 - \Delta\,\eta) = \psi(1) - \Delta\,\eta\psi'(1).$$

The results thus obtained are given in Table I, and shown in Figs. 2 and 3. Figures 2 and 3 show that the solution curve for h is almost linear in the whole range of $0 \leqslant \eta \leqslant 1$ while those for f, g, and j are not so. The values of f, g, h and j at $\eta = 0$ ($X = 0$) give the situation at the moment when shock reaches the boundary $x = 0$.

On putting $\eta = 0$ in the relations (2.8), we get the velocity, pressure, density, and magnetic field distributions u_1, p_1, ρ_1, h_1 behind the shock at $t = 0$, as follows :

$$\left.\begin{aligned} u_1 &= U(x)\, f(0) = f(0)\, k_2 x^{-\lambda} \\ p_1 &= \rho_0(x)\, U^2(x)\, g(0) = g(0)\, k_1 k_2^2\, x^{\alpha-2\lambda} \\ \rho_1 &= \rho_0(x)\, h(0) = h(0)\, k_1 x^{\alpha} \\ h_1 &= \rho_0^{1/2}(x)\, U(x)\, j(0) = j(0)\, k_1^{1/2}\, k_2 x^{(\alpha-2\lambda)/2}. \end{aligned}\right\} \quad \ldots(2.14)$$

TABLE I

η	$f(\eta)$	$g(\eta)$	$h(\eta)$	$j(\eta)$
1	0.769	0.769	4.333	0.433
0.8	0.740	0.783	6.944	0.406
0.6	0.707	0,772	9.248	0.393
0.4	0.682	0.799	11.487	0.391
0.2	0.661	0.829	13.705	0.393
0	0.643	0.865	15.926	0.396

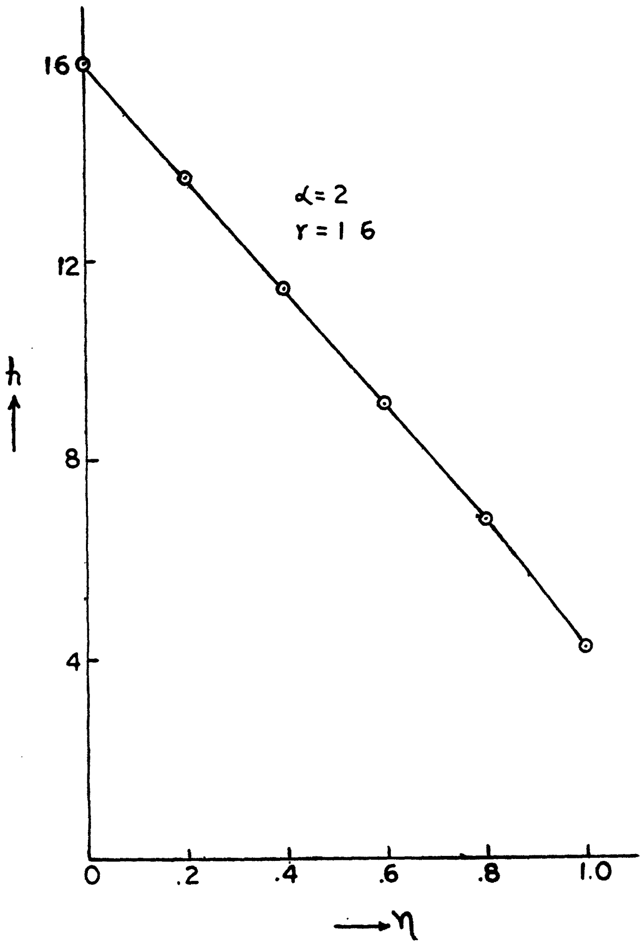

FIG. 2.

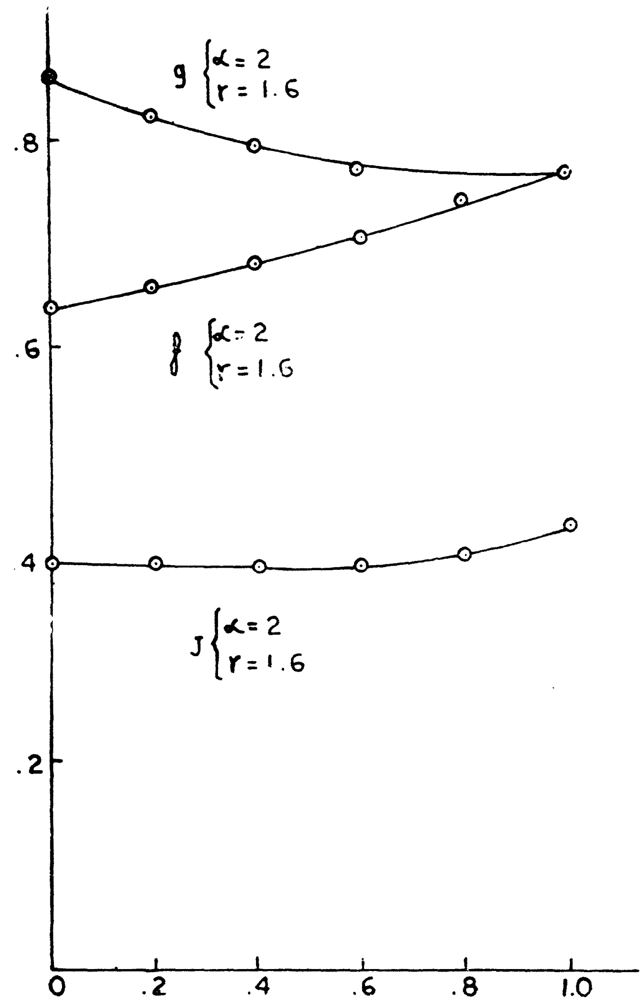

FIG. 3.

From (2.14) we see that ρ_1 is proportional to the original density ρ_0 and u_1 is proportional to $U(x)$. Since $\lambda > 0$ and $(\alpha - 2\lambda) > 0$, according to the numerical value chosen, the pressure, density and magnetic field become zero at $x = 0$ while $u_1 \to \infty$ as $x \to 0$. However, the momentum $\rho_1 u_1$, the kinetic energy per unit mass $\frac{1}{2} \rho_1 u_1^2$, the internal energy and magnetic energy per unit mass become zero.

3. The Second Stage

In this stage, we study the motion of the gas expanding into the vacuum with the initial condition as given by the equations of the final moment of the first stage. In this case it is more convenient to use the Lagrangian co-ordinate a which coincides with our Eulerian co-ordinate $x = x(a, t)$ at $t = 0$. The initial situations are then obtained from (2.14) as

$$\left.\begin{aligned} u(a, 0) &\equiv u_1 = U(a)\, f(0) = f(0)\, k_2\, a^{-\lambda} \\ p(a, 0) &\equiv p_1 = \rho_0(a)\, U^2(a)\, g(0) = g(0)\, k_1 k_2^2\, a^{\alpha-2\lambda} \\ \rho(a, 0) &\equiv \rho_1 = \rho_0(a)\, h(0) = h(0)\, k_1 a^{\alpha} \\ h(a, 0) &\equiv h_1 = \rho_0^{1/2}(a)\, U(a)\, j(0) = j(0)\, k_1^{1/2}\, k_2 a^{(\alpha-2\lambda)/2} \end{aligned}\right\} \quad ...(3.1)$$

for $a \geqslant 0$. Also $u_1 = p_1 = \rho_1 = h_1 = 0$ for $a < 0$ and $x(a, 0) = x_1 = a$.

The equations of motion in Lagrangian form are

$$\frac{\partial x}{\partial a} = \frac{\rho_1}{\rho} \quad ...(3.2)$$

$$\frac{\rho_1}{\rho}\frac{\partial h}{\partial t} + \frac{h\partial u}{\partial a} = 0 \quad ...(3.3)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_1}\frac{\partial p}{\partial a} - \frac{h}{\rho_1}\frac{\partial h}{\partial a} + F_0 \quad ...(3.4)$$

$$u = \frac{\partial x}{\partial t} \quad ...(3.5)$$

$$p\,.\,\rho^{-\gamma} = p_1\rho_1^{-\gamma} = k_1^{1-\gamma}\, k_2^2\, g(0)\, h(0)^{-\gamma} a^{\alpha(1-\gamma)-2\lambda}. \quad ...(3.6)$$

As before, the gas has been assumed to be polytropic. There does not occur a shock wave at $t > 0$, so that the value of $p\,.\,\rho^{-\gamma}$ is obtainable directly from the initial situation ($t = 0$) for the second stage. Also since the body force F_0 cannot be overlooked for large t, we retain it in this case but assume it to be constant so that a similarity solution can still be found for eqns. (3.2) – (3.6) under initial conditions (3.1). We have thus,

$$\left.\begin{aligned} x &= \tfrac{1}{2}\, t^2 F_0 + a\,.\,r(\xi) \\ u &= tF_0 + F(\xi)\, u_1(a) \\ p &= G(\xi)\, p_1(a) \\ \rho &= H(\xi)\, \rho_1(a) \\ h &= J(\xi)\, h_1(a) \end{aligned}\right\} \quad ...(3.8)$$

where

$$\xi = \frac{U(a)}{a}\, t \text{ varies as } a^{-\lambda-1}t, \quad (-\infty \leqslant \xi \leqslant 0) \quad ...(3.9)$$

and the shock velocity is given by

$$U(a) = k_2 a^{-\lambda}. \quad ...(3.10)$$

It can be easily verified that the special type of similarity solution (3.8) chosen to fit the singularity at $a = 0$, satisfies the initial conditions (3.1) if

$$r(0) = F(0) = G(0) = H(0) = J(0) = 1. \qquad \text{...(3.11)}$$

Substituting (3.8) into (3.2) - (3.6), we have the following equations for r, F, G, H, and J.

$$r(\xi) - (1 + \lambda)\,\xi r' = \frac{1}{H(\xi)} \qquad \text{...(3.12}a\text{)}$$

$$\frac{1}{H(\xi)}\,J'(\xi) = f(0)\,J(\xi)\,\{(1 + \lambda)\,\xi F'(\xi) + \lambda F(\xi)\} \qquad \text{...(3.12}b\text{)}$$

$$f(0)\,h(0)\,F'(\xi) = (1 + \lambda)\,\xi\{g(0)\,G'(\xi) + J^2(0)\,J'(\xi)\} - \left\{(\alpha - 2\lambda)\,g(0)\,G(\xi) + \left(\frac{\alpha}{2} - \lambda\right)J^2(0)\,J(\xi)\right\} \qquad \text{...(3.12}c\text{)}$$

$$r'(\xi) = f(0)\,F(\xi) \qquad \text{...(3.12}d\text{)}$$

$$G(\xi)\,H^{-\gamma}(\xi) = 1. \qquad \text{...(3.12}e\text{)}$$

Since from the first stage equations, λ, $f(0)$, $g(0)$, $h(0)$ and $J(0)$ are known, the eqn. (3.12) can be integrated numerically from $\xi = 0$ to $\xi = -\infty$. For this we eliminate $r(\xi)$ and $H(\xi)$ to obtain a modified form,

$$\frac{G^{-(1/\gamma)-1}}{\gamma f(0)}\,G' = (1 + \lambda)\,\xi F' + \lambda F(\xi) \qquad \text{...(3.13}a\text{)}$$

$$\frac{G(\xi)^{-(1/\gamma)}}{f(0)\,J(\xi)}\,J'(\xi) = (1 + \lambda)\,\xi F'(\xi) + \lambda F(\xi) \qquad \text{...(3.13}b\text{)}$$

$$f(0)\,h(0)\,F'(\xi) = (1 + \lambda)\,\xi\{g(0)\,G'(\xi) + J^2(0)\,J'(\xi)\} - \left\{(\alpha - 2\lambda)\,g(0)\,G(\xi) + \left(\frac{\alpha}{2} - \lambda\right)J^2(0)\,J^2(\xi)\right\}. \qquad \text{...(3.13}c\text{)}$$

By carefully observing the eqns. (3.13a) and (3.13b), one can easily verify that the field is 'frozen-in', an assumption with which we have started initially. We now seek a solution of (3.13) under the conditions :

$$F(0) = G(0) = J(0) = 1. \qquad \text{...(3.14)}$$

The functions $r(\xi)$ and $H(\xi)$ are given respectively by

$$H = G^{1/\gamma}, \quad r = 1 + f(0)\int_0^{\xi} F(\xi)\,d\xi. \qquad \text{...(3.15)}$$

Numerical integration of the eqn. (3.13) with (3.14) is performed up to $\xi = -4$ taking $\lambda = 0.435$ and $f(0)$, $g(0)$, $h(0)$ and $j(0)$ from Table I. The functions $r(\xi)$ and $H(\xi)$ are obtained from (3.15). The results are given in Table II and shown in Figs. 4, 5 and 6.

TABLE II

ξ	$F(\xi)$	$J(\xi)$	$G(\xi)$	$H(\xi)$	$r(\xi)$
0	1	1	1	1	1
− 1	1.1041	0.7203	0.5525	0.69007	0.2901
− 2	1.1946	0.5253	0.3131	0.48395	− 0.5362
− 3	1.2596	0.4098	0.2030	0.3691	− 1.4297
− 4	1.3062	0.3370	0.1453	0.2995	− 2.3559

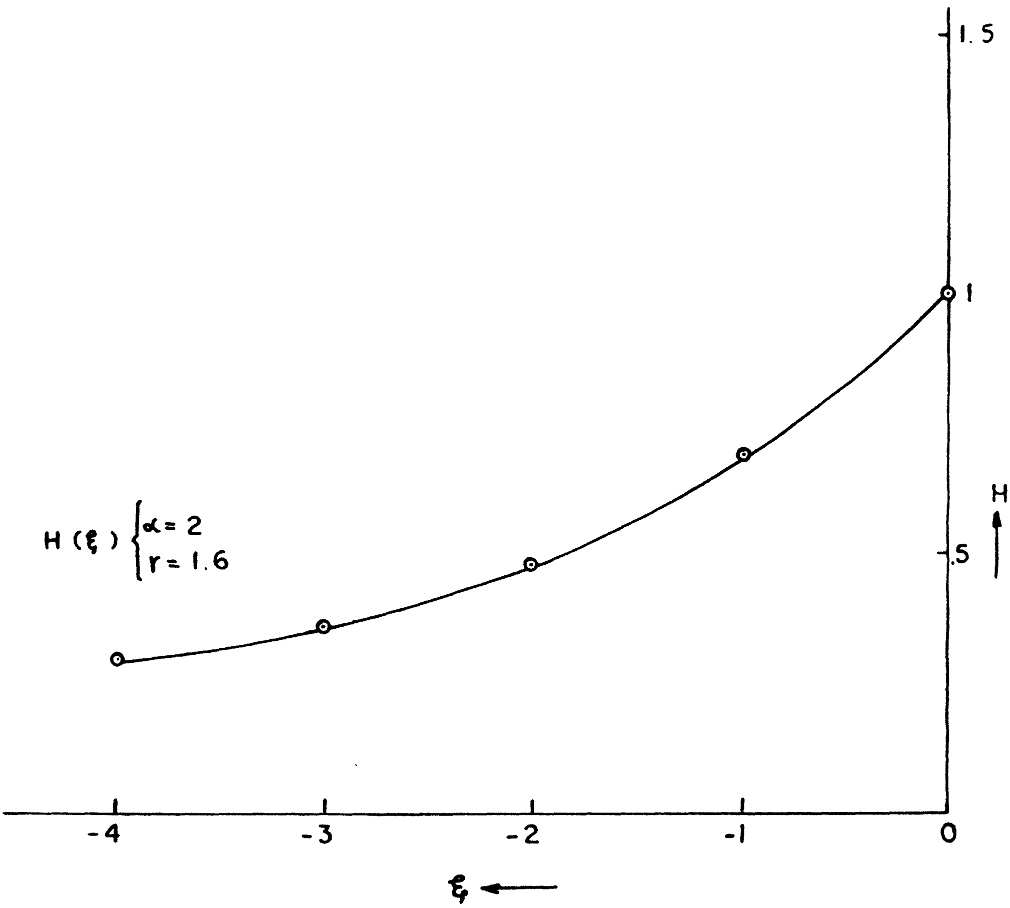

FIG. 4.

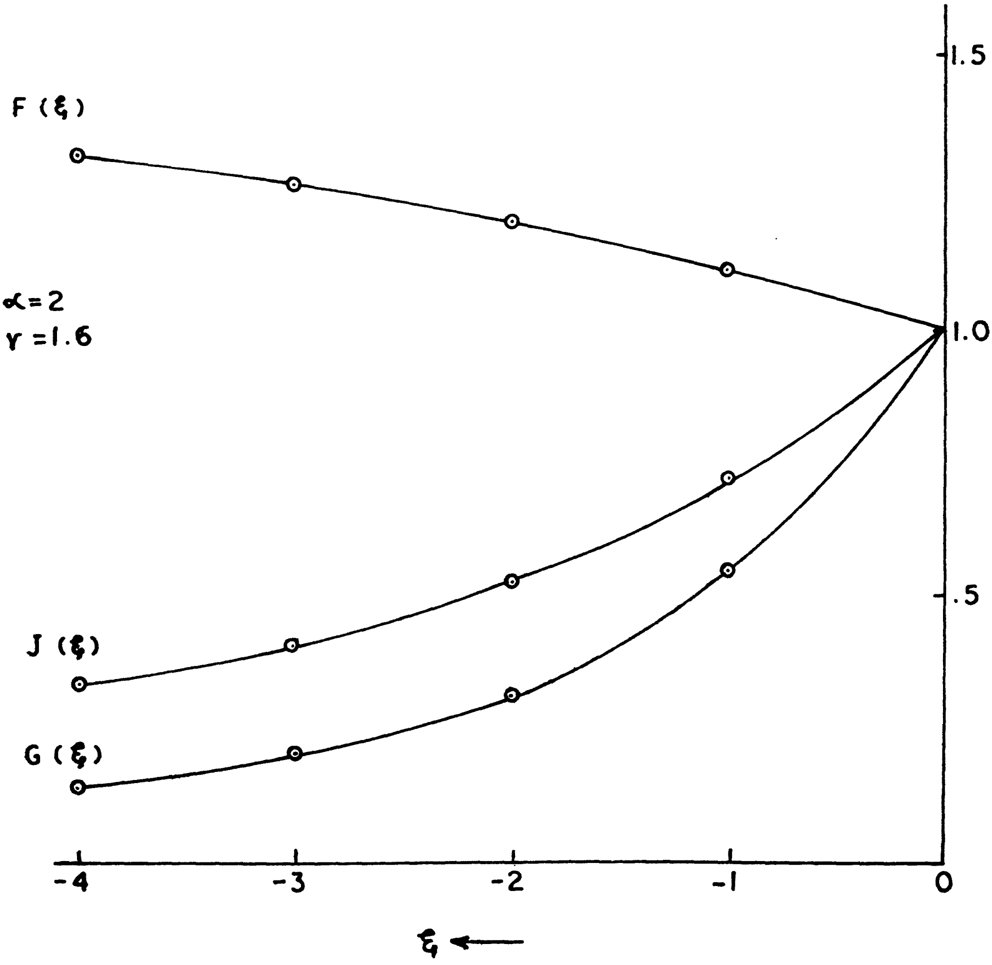

FIG. 5.

The curves for $r(\xi)$, $F(\xi)$, $G(\xi)$, $H(\xi)$ and $J(\xi)$ are not almost linear as were those for $f(\eta)$, $g(\eta)$, $h(\eta)$ and $J(\eta)$.

Sakurai (1960) has calculated the values of λ for $\alpha = 2$, 1 and $\frac{1}{2}$ in the case of different gases. We have here taken the value $\lambda = 0.435$ corresponding to $\alpha = 2$ and $\gamma = \frac{5}{3}$ together with an aligned magnetic field for numerical calculations. Many investigations on the propagation of shock waves through non-uniform medium have been carried out in the case of $\alpha < 0$ e.g. Sedov (1959) but it should be noted that ρ_0 in the case of $\alpha < 0$ has no edge and so our problem remains different from the one treated in Sedov (1959). This problem may be related to the situation that may arise with shock waves in stars where the details of reflection process might be needed in the study of nova phenomenon.

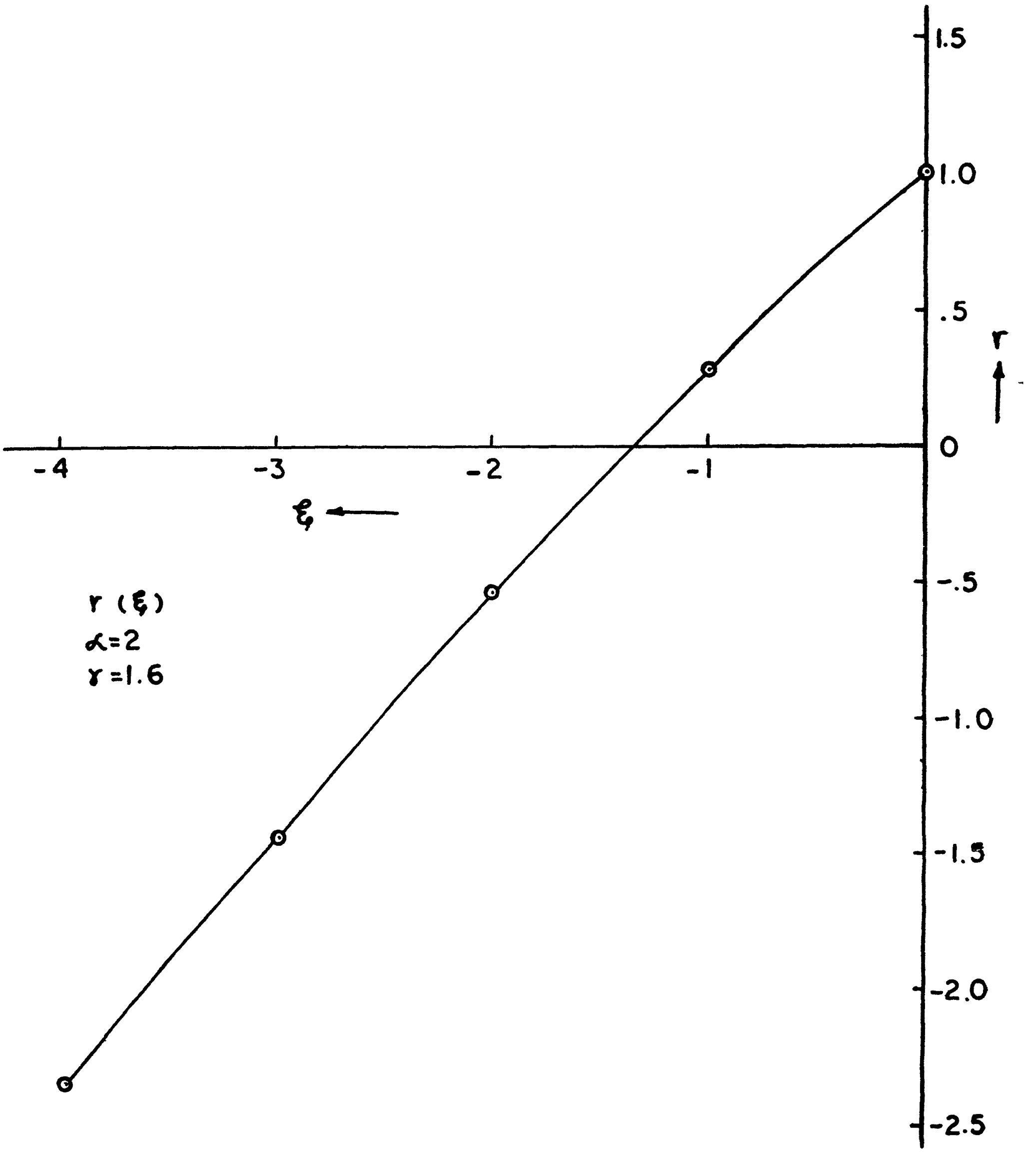

Fig. 6.

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